CHAPTER 10

METRIC SPACES

● PROBLEM 10-1

Let R represent the set of real numbers. Show that one of the possible metrics for R is the absolute value

$$d(x, y) = |x - y| \tag{1}$$

SOLUTION:

DEFINITION OF A METRIC SPACE

Let X be a set. Function $d(x,y) \in R$

$$d : X \times X \to R \tag{2}$$

is said to be a metric on X if for all $x, y, z \in X$

1. $d(x, y) \geq 0$.
2. $d(x, y) = 0$ iff $x = y$.
3. $d(x, y) = d(y, x)$.
4. $d(x, z) \leq d(x, y) + d(y, z)$.

The set X with a metric d is called a metric space and is denoted by (X, d). The function $d(x, y)$ is called the distance between x and y.

■

For any real numbers x, y, and z we have

$$|x - y| \geq 0$$

$$|x - y| = 0 \quad \text{iff} \quad x - y = 0 \quad \text{iff} \quad x = y$$

$$|x - y| = |y - x|$$

$$|x - z| \leq |x - y| + |y - z|.$$

Hence (1) defines a metric.

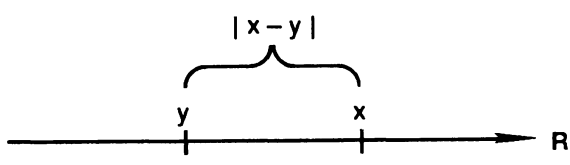

FIGURE 1

If x and y are real numbers on the real axis, then $|x-y|$ is the distance between them.

● PROBLEM 10-2

1. Is it possible to define a metric on any set?

2. Show that the set R^n with the distance defined by

$$d(x, y) = \max \{|x_i - y_i|, 1 \leq i \leq n\} \tag{1}$$

is a metric space.

SOLUTION:

1. Let X be any set. Then, function d defined by

$$d(x,y) = \begin{cases} 1 & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases} \tag{2}$$

is a metric. Therefore a metric may be defined for any set.

2. We have

$$d(x, y) \geq 0 \qquad \text{because } |x_i - y_i| \geq 0$$

$$0 = d(x, y) = \max \{|x_i - y_i|, 1 \leq i \leq n\} \text{ iff}$$

$$|x_i - y_i| = 0 \text{ for } 1 \leq i \leq n \text{ iff } x_i = y_i \text{ for}$$

$$1 \leq i \leq n \text{ iff } x = y.$$

We denote

$$x = (x_1, \ldots, x_n)$$

$$y = (y_1, \ldots, y_n) \tag{3}$$

Symmetry is obvious

$$d(x, y) = d(y, x) \tag{4}$$

Triangle inequality.

Suppose for some $1 \leq l \leq n$

$$d(x, y) = |x_i - y_i|$$

we have

$$d(x, y) = | x_i - y_i | \leq | x_l - z_l | + | z_l - y_l | \leq$$

$$\leq d(x, z) + d(z, y). \tag{5}$$

● PROBLEM 10-3

Show that in the set R^n,

$$d_p(x, y) = \sqrt[p]{\sum_1^n |x_i - y_i|^p} \tag{1}$$

is a metric for each $p \geq 1$.

SOLUTION:

We shall prove the triangle inequality.

For $p \geq 1$, the function

$$f(x) = x^p \tag{2}$$

for $x \in R^1$, $x \geq 0$ satisfies

$$f''(x) \geq 0 \tag{3}$$

and hence it is convex. Thus, $f(x)$ satisfies

$$f(\alpha x + [1 - \alpha]y) \leq \alpha f(x) + (1 - \alpha) f(y) \tag{4}$$

for $x, y \geq 0$.

For any set of real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ let

$$A = \sqrt[p]{\sum_1^n |a_i|^p} \qquad B = \sqrt[p]{\sum_1^n |b_i|^p} \tag{5}$$

Define x, y, and α as; $\forall i \in [1, n]$

$$x_i = \frac{a_i}{A}, \qquad y_i = \frac{b_i}{B}, \qquad \alpha_i = \frac{A}{A + B}. \tag{6}$$

Substituting (6) for each $i = 1, 2, \ldots, n$ into (4) and adding all n inequalities, we obtain

$$\sqrt[p]{\sum_1^n |a_i - b_i|^p} \leq \sqrt[p]{\sum_1^n |a_i|^p} + \sqrt[p]{\sum_1^n |b_i|^p} \tag{7}$$

Equation (7) is called Minkowski's inequality.

Substituting

$$a_i = x_i - y_i$$

$$b_i = y_i - z_i \tag{8}$$

for $n = 1, 2, \ldots, n$ into (7) we obtain

$$d_p(x, z) \le d_p(x, y) + d_p(y, z). \tag{9}$$

In particular for $n = 2$ and $p = 2$, we obtain

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \tag{10}$$

which is a distance on R^2.

• PROBLEM 10-4

Let d be a metric on a set X. Show that

$$d_0(x,y) = \frac{d(x,y)}{1 + d(x,y)} \tag{1}$$

where $x, y \in X$ is also a metric on X.

SOLUTION:

We have to show that d_0 satisfies the triangle inequality. All remaining conditions for a metric are obviously satisfied.

Since d is a metric we have

$$\frac{d(x,y)}{1 + d(x,y) + d(y,z)} \le \frac{d(x,y)}{1 + d(x,y)} = d_0(x,y) \tag{2}$$

$$\frac{d(y,z)}{1 + d(x,y) + d(y,z)} \le \frac{d(y,z)}{1 + d(y,z)} = d_0(y,z) \tag{3}$$

Function d is a metric. Hence $d(x, z) \le d(x, y) + d(y, z)$ and

$$d_0(x,z) = \frac{d(x,z)}{1 + d(x,z)} \le \frac{d(x,y) + d(y,z)}{1 + d(x,y) + d(y,z)} =$$

$$= \frac{d(x,y)}{1 + d(x,y) + d(y,z)} + \frac{d(y,z)}{1 + d(x,y) + d(y,z)} \le d_0(x,y) + d_0(y,z) \tag{4}$$

Thus d_0 is a metric.

Let X represent the set of all functions from the closed interval [0, 1] into itself. Show that for any $f, g \in X$

$$d(f, g) = \text{least upper bound } \{ |f(x) - g(x)| : x \in [0, 1]\} \qquad (1)$$

is a metric.

SOLUTION:

Any subset of the real numbers which has an upper bound has a least upper bound and

$$0 \leq |f(x) - g(x)| \leq 1 \quad \text{for all } f, g \in X \text{ and } x \in [0, 1]. \qquad (2)$$

Hence $d(f, g)$ is defined for all $f, g \in X$.

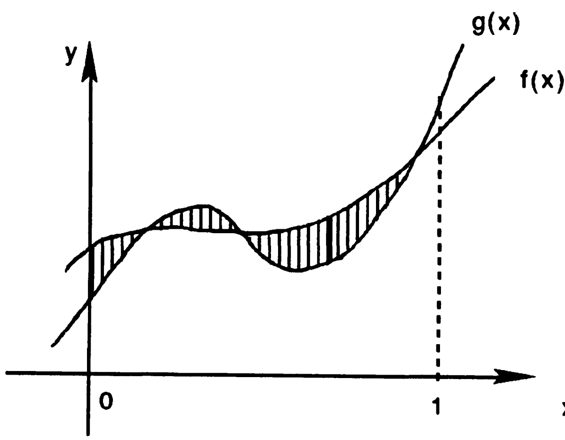

FIGURE 1

1. Since

$$|f(x) - g(x)| \geq 0, x \in [0, 1] \qquad (3)$$

the least upper bound of (3) is non-negative, i.e.,

$$d(f, g) \geq 0 \quad \text{for all } f, g \in X. \qquad (4)$$

2. $$d(f, g) = 0 \quad \text{iff} \quad f = g. \qquad (5)$$

If $d(f, g) = 0$, then a least upper bound

$$\{ |f(x) - g(x)| : x \in [0, 1] \} = 0 \qquad (6)$$

and

$$\{\ |f(x) - g(x)| : x \in [0, 1]\ \} = \{\ 0\ \}. \tag{7}$$

Hence

$$|f(x) - g(x)| = 0 : x \in [0, 1] \tag{8}$$

and $f(x) = g(x)$ for all $x \in [0, 1]$.

If $f = g$, then $f(x) = g(x)$ for all $x \in [0, 1]$ and

$$\{\ |f(x) - g(x)| : x \in [0, 1]\ \} = \{\ 0\ \}. \tag{9}$$

Hence

$$d(f, g) = 0. \tag{10}$$

3. $d(f, g) = d(g, f)$ for all $f, g \in X$ because

$$|f(x) - g(x)| = |g(x) - f(x)|. \tag{11}$$

4.
$$d(f, h) \leq d(f, g) + d(g, h) \tag{12}$$

because for all $f, g, h \in X$

$$|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)| \tag{13}$$

for all $x \in [0, 1]$.

● PROBLEM 10-6

1. Let (X_1, d_1) and (X_2, d_2) represent metric spaces. Show that $(X_1 \times X_2, d)$ where

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2) \tag{1}$$

for $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$, is a metric space.

2. Applying (1), define a metric on a Cartesian product of n metric spaces $X_1 \times X_2 \times \ldots \times X_n$.

SOLUTION:

1. Since d_1 and d_2 are metrics

$$d_1(x_1, y_1) \geq 0 \text{ and } d_2(x_2, y_2) \geq 0 \tag{2}$$

and

$$d((x_1, x_2), (y_1, y_2)) \geq 0$$

for all $x_1, y_1 \in X_1$ and for all $x_2, y_2 \in X_2$.

Similarly since $d_1(x_1, y_1) = d_1(y_1, x_1)$ and $d_2(x_2, y_2) = d(y_2, x_2)$ we have

$$d((x_1, x_2), (y_1, y_2)) = d((y_1, y_2), (x_1, x_2)) \tag{4}$$

$$d((x_1, x_2), (y_1, y_2)) = 0 \quad \text{iff} \quad (x_1, x_2) = (y_1, y_2). \tag{5}$$

Suppose $d((x_1, x_2), (y_1, y_2)) = 0$ then

$$d_1(x_1, y_1) = 0 \text{ and } d_2(x_2, y_2) = 0.$$

Thus

$$x_1 = y_1 \quad \text{and} \quad x_2 = y_2 \text{ or } (x_1, x_2) = (y_1, y_2). \tag{6}$$

If $(x_1, x_2) = (y_1, y_2)$, then

$$d_1(x_1, y_1) = 0 \quad \text{and} \quad d_2(x_2, y_2) = 0. \tag{7}$$

Thus

$$d((x_1, x_2), (y_1, y_2)) = 0. \tag{8}$$

Now, we will show that

$$d((x_1, x_2), (z_1, z_2)) \le d((x_1, x_2), (y_1, y_2)) + d((y_1, y_2), (z_1, z_2)). \tag{9}$$

Indeed, since d_1 and d_2 are metrics

$$d((x_1, x_2), (z_1, z_2)) = d_1(x_1, z_1) + d_2(x_2, z_2) \le$$

$$\le d_1(x_1, y_1) + d_1(y_1, z_1) + d_2(x_2, y_2) + d_2(y_2, z_2) =$$

$$= d((x_1, x_2), (y_1, y_2)) + d((y_1, y_2), (z_1, z_2)). \tag{10}$$

2. If $(X_1, d_1), (X_2, d_2), \ldots, (X_n, d_n)$ are metric spaces, then

$$d(x, y) = d_1(x_1, y_1) + \ldots + d_n(x_n, y_n) \tag{11}$$

is a metric on $X_1 \times X_2 \times \ldots \times X_n$.

● PROBLEM 10–7

The following metrics are defined on R^2

$$d_1((x_1,y_1),(x_2,y_2)) = \begin{cases} 0 & \text{if } (x_1,y_1) = (x_2,y_2) \\ 1 & \text{if } (x_1,y_1) \ne (x_2,y_2) \end{cases}$$

$$d_2((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$d_3((x_1, y_1), (x_2, y_2)) = | x_1 - x_2 | + | y_1 - y_2 |$$

$$d_4((x_1, y_1), (x_2, y_2)) = \max \left(| x_1 - x_2 |, | y_1 - y_2 | \right).$$

Find the ball of radius 1 and center (0, 0) with respect to metrics d_1, d_2, d_3, and d_4.

SOLUTION:

DEFINITION OF A BALL

The convex set

$$B_d(a, r) = \{x : d(x, a) < r\}$$

is called an open ball of radius r and center a.

We have $B_{d_1}((0,0),1) = \{(0,0)\}$.

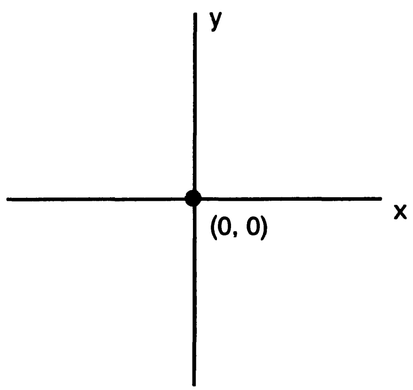

FIGURE 1

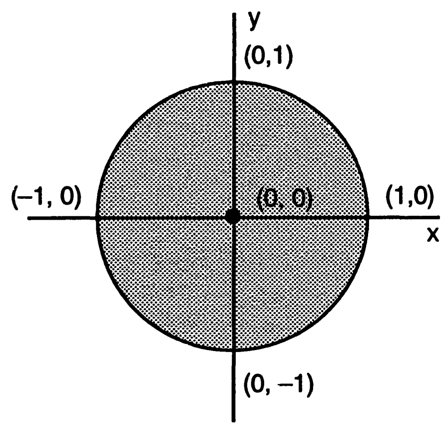

FIGURE 2

Figure 2 illustrates the d_2-ball, $B_{d_2}((0,0), 1)$.

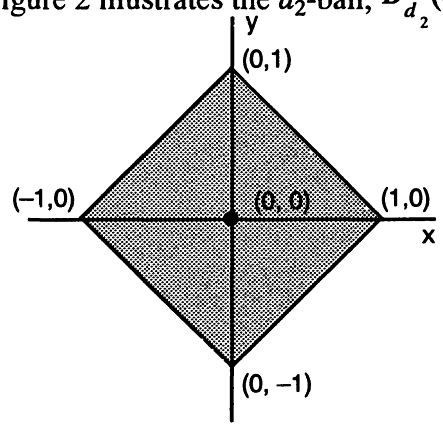

FIGURE 3

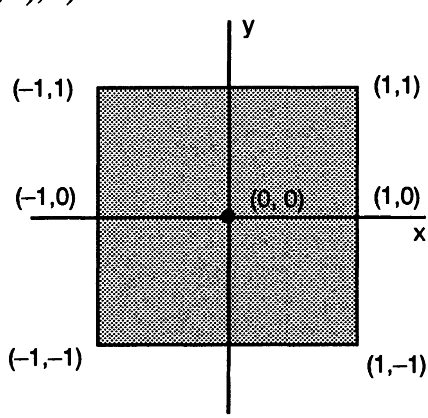

FIGURE 4

B_{d_3} is shown in Figure 3.

B_{d_4} is shown in Figure 4.

Sometimes when it is obvious what metric is used we shall write $B(a, r)$ instead of $B_d(a, r)$.

● PROBLEM 10–8

1. Describe explicitly the ball $B(x_0, r)$ in the metric space (R, d) where $d(x, y) = |x - y|$.

2. Let $C[0, 1]$ be the collection of all continuous functions on $[0, 1]$ with the metric defined by

$$d(f, g) = \sup \{ |f(x) - g(x)| : x \in [0, 1] \}. \qquad (1)$$

Find the ball $B(f_0, \varepsilon)$ where f_0 is a continuous function and $\varepsilon > 0$.

SOLUTION:

1. The ball $B(x_0, r)$ is shown in Figure 1.

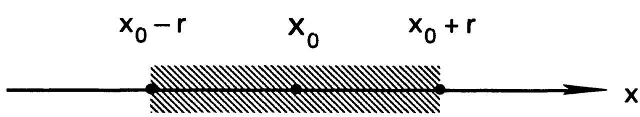

FIGURE 1

The ball consists of all the points x on the real axis such that

$$|x - x_0| < r \quad \text{or} \quad x_0 - r < x < x_0 + r. \qquad (2)$$

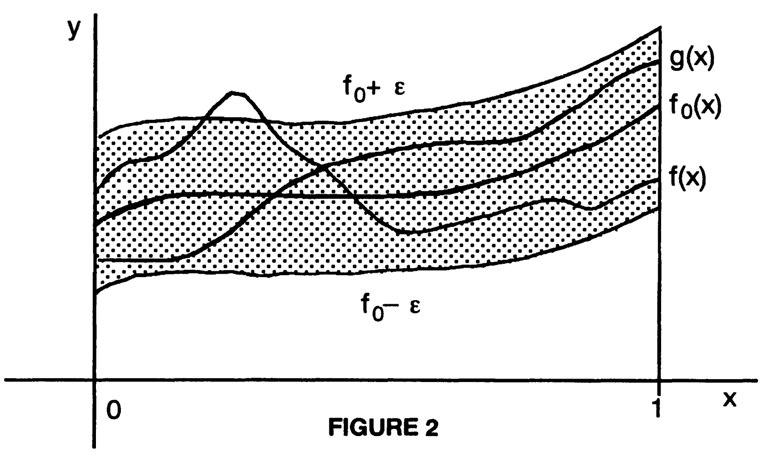

FIGURE 2

2. Let f_0 be a continuous function as given in Figure 2. The open ball $B(f_0, \varepsilon)$ consists of all continuous functions which lie in the area bounded by, (and excluding), the lines at $f_0 - \varepsilon$ and $f_0 + \varepsilon$.

● PROBLEM 10-9

Prove this theorem:

THEOREM

Let $B(x_0, r)$ be a ball. For every point $y \in B(x_0, r)$, there exists a ball $B(y, r')$ such that

$$B(y, r') \subset B(x_0, r). \tag{1}$$

■

SOLUTION:

Point $y \in B(x_0, r)$ hence

$$d(x_0, y) < r. \tag{2}$$

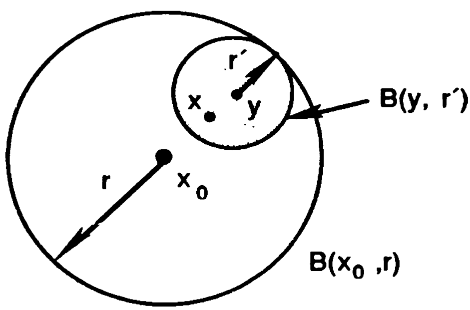

FIGURE 1

Let us define

$$r' = r - d(x_0, y) > 0. \tag{3}$$

We will show that

$$B(y, r') \subset B(x_0, r). \tag{4}$$

Suppose

$$x \in B(y, r') \tag{5}$$

then

$$d(x, y) < r' \tag{6}$$

or

$$d(x, y) < r - d(x_0, y)$$

$$d(x, y) + d(x_0, y) < r. \tag{7}$$

But d is a metric, so

$$d(x, x_0) \leq d(x, y) + d(y, x_0) < r. \tag{8}$$

Hence

$$d(x, x_0) < r$$

and

$$x \in B(x_0, r). \tag{9}$$

● PROBLEM 10–10

1. Prove that if B_1 and B_2 are balls with the same center, then one of them is a subset of the other.

2. Prove this theorem:

THEOREM

Let B_1 and B_2 be balls and let $x \in B_1 \cap B_2$. Then, a ball B exists such that

$$x \in B \subset B_1 \cap B_2 \tag{1}$$

■

SOLUTION:

1. Let $B_1(x_0, r_1)$ and $B_2(x_0, r_2)$. Since r_1 and r_2 are real numbers, we have either $r_1 \leq r_2$ or $r_2 \leq r_1$. Suppose $r_1 \leq r_2$. Then if $x \in B_1(x_0, r_1)$ we obtain $d(x_0, x) < r_1 \leq r_2$.

Thus $d(x_0, x) < r_2$ and $x \in B_2(x^0, r_2)$.

If $r_2 \leq r_1$, then $B_2(x_0, r_2) \subset B_1(x_0, r_1)$.

2. Let x be a point, such that

$$x \in B_1 \cap B_2. \tag{2}$$

Then

$$x \in B_1 \quad \text{and} \quad x \in B_2.$$

By Problem 10–9, if $x \in B_1(x_0, r_1)$, then a ball $B'_1(x, r'_1)$ exists such that

$$x \in B'_1 \subset B_1. \tag{3}$$

Similarly, a ball $B'_2(x, r'_2)$ exists such that

$$x \in B'_2 \subset B_2 \tag{4}$$

as shown in Figure 1.

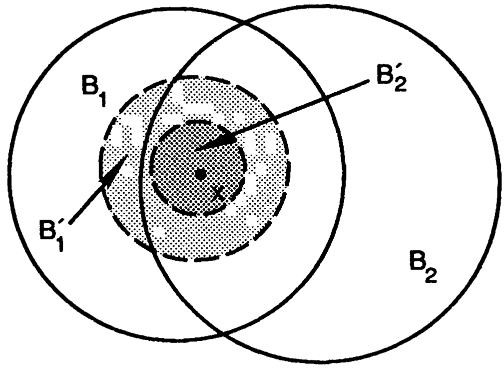

FIGURE 1

The balls B'_1 and B'_2 both have the same center. By part 1 of this problem, one of them is contained in the other. Suppose

$$B'_2 \subset B'_1. \tag{5}$$

Then

$$x \in B'_2 \subset B'_1 \subset B_1. \tag{6}$$

Let $B = B'_2$, then

$$x \in B \subset B_1 \cap B_2. \tag{7}$$

● PROBLEM 10–11

1. Let R be the set of real numbers with the absolute value metric. Show that the open interval (0, 1) is an open set.

2. Show that the ball

$$B(x_0, r) = \{ x : d(x, x_0) < r \} \tag{1}$$

is an open set.

SOLUTION:

DEFINITION OF AN OPEN SET

Let (X, d) be a metric space. A subset $A \subset X$ is said to be open, if for every $x \in A$, a ball $B(x, r)$ exists such that

$$x \in B(x, r) \subset A.$$

■

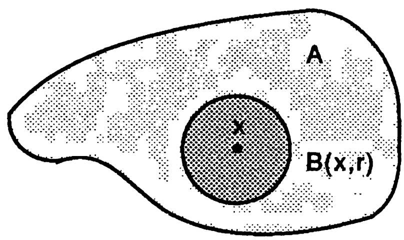

FIGURE 1

1. Let $x \in (0, 1)$, then

$$0 < x < 1. \tag{2}$$

We define

$$r = \min \{ x, 1 - x \}. \tag{3}$$

Then

$$B(x, r) \subset (0, 1) \tag{4}$$

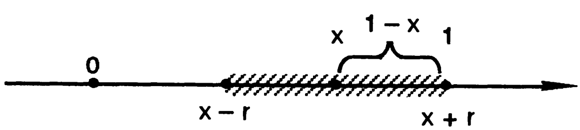

FIGURE 2

Any open interval (a, b) is an open set.

2. Consider ball (1). Let

$$y \in B(x_0, r). \tag{5}$$

Then $d(x_0, y) < r$.

By virtue of Problem 10–9, we conclude that a ball $B'(y, r')$ exists such that

$$y \in B'(y, r') \subset B(x_0, r) \tag{6}$$

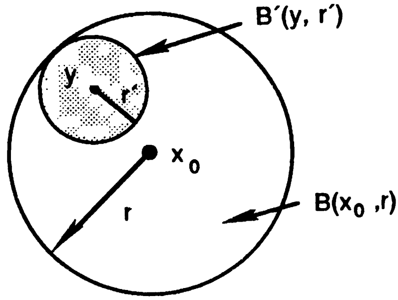

FIGURE 3

● PROBLEM 10–12

> Prove the following:
> **THEOREM**
>
> Let (X, d) be a metric space. Then
>
> 1. X, ϕ are open sets,
> 2. the intersection of any two open sets is an open set,
> 3. the union of any family of open sets is an open set.
>
> ■

SOLUTION:

1. If $x \in X$ and $r > 0$, then

$$B(x, r) \subset X. \tag{1}$$

Hence X is an open set.

For each $x \in \phi$ and any $r > 0$,

$$B(x, r) \subset \phi. \tag{2}$$

Hence ϕ is an open set.

2. Suppose A and B are open subsets of (X, d) and $x \in A \cap B$. Since both A and B are open, balls exist such that

$$B_1(x, r_1) \subset A$$

$$B_2(x, r_2) \subset B. \tag{3}$$

Setting

$$r = \min(r_1, r_2) \tag{4}$$

we obtain

$$B(x, r) \subset A \cap B. \tag{5}$$

Hence $A \cap B$ is an open set.

3. Let $\bigcup_{\alpha} A_\alpha$ be a union of a family of open sets. Suppose

$$x \in \bigcup_{\alpha} A_\alpha \tag{6}$$

Then an open set A_α exists such that is a member of this family and

$$x \in A_\alpha \tag{7}$$

and since A_α is open

$$x \in B(x,r) \subset A_\alpha \subset \bigcup_{\alpha} A_a \tag{8}$$

● PROBLEM 10–13

> Let (X, d) be a metric space. Show that a subset $A \subset X$ is open if and only if A is the union of a family of balls.

SOLUTION:

We shall prove

$$(A \text{ is open}) \Leftrightarrow (A = \bigcup_{i \in I} B_i) \tag{1}$$

$\Leftarrow$ In Problem 10–12, we proved that the union of any family of open sets is an open set. In Problem 10–11, we proved that a ball is an open set. Hence $A = \bigcup_{i \in I} B_i$ is an open set.

$\Rightarrow$ Suppose A is an open set. Then

$$\forall x \in A \quad \exists B(x, r) \subset A. \tag{2}$$

Therefore

$$\bigcup_{x \in A} B(x,r) \subset A \tag{3}$$

But the ball $B(x, r) \subset A$ exists for every $x \in A$. Hence

$$A = \bigcup_{x \in A} B(x, r). \tag{4}$$

● PROBLEM 10-14

Show that the closed interval [0, 1] is a closed subset of R with the absolute value metric.

SOLUTION:

DEFINITION OF A CLOSED SET

Let (X, d) be a metric space. A subset B of X is said to be closed in (X, d) when its complement $X - B$ is an open set in (X, d).

■

We shall show that the set $R - [0, 1]$ is open in R.

Suppose

$$x \in R - [0, 1] \tag{1}$$

Then, either $x > 1$ or $x < 0$. We can assume that $x > 1$, (the proof for $x < 0$ is similar).

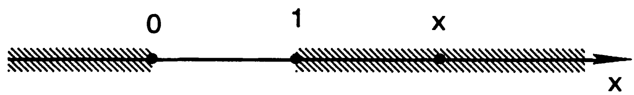

FIGURE 1

We have

$$0 < \delta = x - 1 \tag{2}$$

and

$$B(x, {}^{\delta}/_{2}) \subset R - [0, 1]. \tag{3}$$

Therefore $R - [0, 1]$ is an open set and $[0, 1]$ is a closed set.

Similarly we can show that a closed ball in the metric space (X, d)

$$B(x, r) = \{ y : d(x, y) \leq r \} \tag{4}$$

is a closed set.

● PROBLEM 10-15

Prove the following:

THEOREM

Let (X, d) be a metric space. Then

1. X and ϕ are closed subsets of X,

2. the union of two closed sets, hence of a finite number of such sets, is a closed set,

3. the intersection of any family of closed sets is a closed set.

■

SOLUTION:

1. ϕ is an open set, therefore $X - \phi = X$ is closed, because it is a complement of an open set. X is an open set, therefore ϕ is a closed set because $\phi = X - X$.

2. Suppose B_1 and B_2 are closed sets. Then $X - B_1$ and $X - B_2$ are open sets. We have

$$X - (B_1 \cup B_2) = (X - B_1) \cap (X - B_2) \tag{1}$$

Since $X - B_1$ and $X - B_2$ are open sets, $(X - B_1) \cap (X - B_2)$ is an open set. Hence $X - (B_1 \cup B_2)$ is a closed set.

3. We shall show that

$$\bigcap_\alpha B_\alpha \tag{2}$$

is a closed set when all B_α's are closed sets.

$$X - \left(\bigcap_\alpha B_\alpha\right) = \bigcup_\alpha (X - B_\alpha) \tag{3}$$

Suppose all sets B_α are closed. Then all sets $X - B_\alpha$ are open sets and their union $\bigcup_\alpha (X - B_\alpha)$ is an open set.

Therefore $X - \left(\bigcap_\alpha B_\alpha\right)$ is an open set and $\bigcap_\alpha B_\alpha$ is a closed set.

● PROBLEM 10-16

Which of the following subsets of (R^2, d) where d is the Pythagorean metric are closed?

1. $\{(x, y) : x = 2\}$.

2. $\{(x, y) : x, y \text{ are integers}\}$.

3. $\{(x, y) : y = x^2\}$.

4. $\{(x, y) : x^2 + y^2 \leq 1 \text{ and } x < 1\}$.

5. $\{(x, y) : x^2 + y^2 > 2\}$.

SOLUTION:

1.

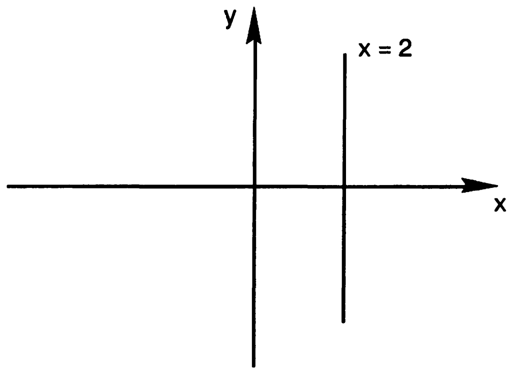

FIGURE 1

This is a closed set because $R^2 - \{(x, y) : x = 2\}$ is open.

2.

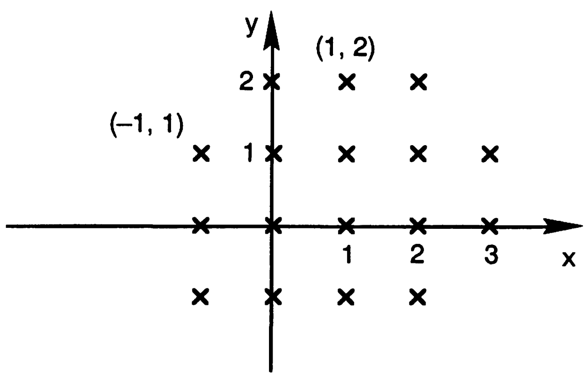

FIGURE 2

This set is closed.

3.

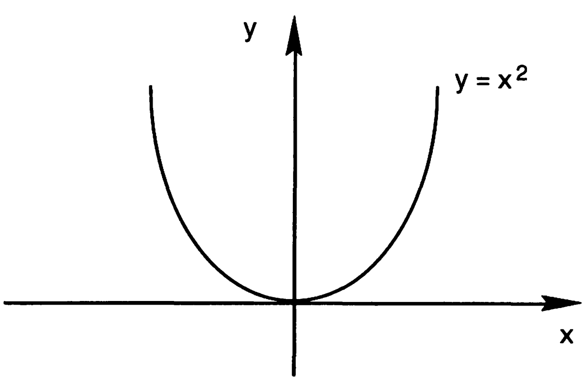

FIGURE 3

This set is closed.

4.

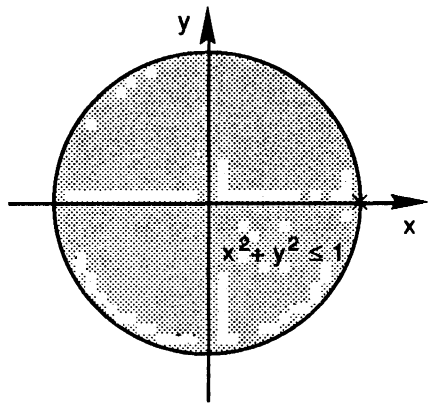

FIGURE 4

This set is neither closed nor open.

5.

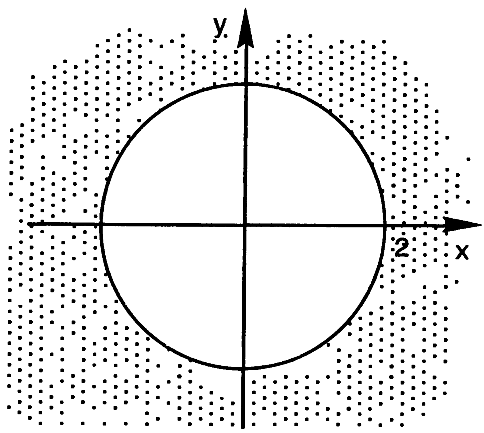

FIGURE 5

Since the circle $x^2 + y^2 = 2$ is not in the set, then the set is open..

● PROBLEM 10-17

DEFINITION OF CONVERGENCE

A sequence of points (x_n) of a metric space (X, d) is convergent to the point x of this space, if the sequence of real numbers $d(x_n, x)$ is convergent to zero. The point $x \in X$ is called the limit of the sequence (x_n). We write

$$x = \lim_{n \to \infty} x_n \tag{1}$$

■

Write down this definition using the symbolism of logic.

Given an example to illustrate that a given sequence can be convergent with respect to one metric and not convergent with respect to another metric.

SOLUTION:

The definition of a limit point may be written in symbolism of logic as follows:

$$\left(\lim_{n\to\infty} x_n = x\right) \equiv \left(\lim_{n\to\infty} d(x_n, x) = 0\right) \equiv$$
$$\equiv \forall \varepsilon > 0 \quad \exists k \in N \quad \forall n \in N$$
$$(n > k) \Rightarrow (d(x_n, x) < \varepsilon) \tag{2}$$

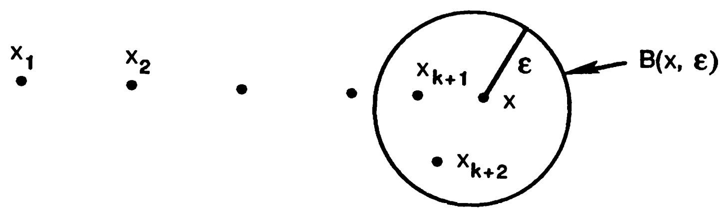

FIGURE 1

If $\lim_{n\to\infty} x_n = x$, then for any ball $B(x, \varepsilon)$ all but a finite number of elements of (x_n) are located inside the ball $B(x, \varepsilon)$.

Suppose the set of real numbers R is equipped with the metric $d_1(x, y) = |x - y|$ and with the metric

$$d_2(x,y) = \begin{cases} 0 & \text{when } x = y \\ 1 & \text{when } x \neq y \end{cases}$$

The sequence $({}^1/_n)$ is given. In the space (R, d_1), this sequence is convergent. In the space (R, d_2), this sequence is not convergent.

● PROBLEM 10–18

Is it possible for a convergent sequence to have two different limits?

SOLUTION:

Let (X, d) be a metric space. Suppose sequence (X_n) has two different limits x_1 and x_2, $x_1 \neq x_2$. Then

$$\lim_{n \to \infty} x_n = x_1 \tag{1}$$

$$\lim_{n \to \infty} x_n = x_2 \tag{2}$$

Since $x_1 \neq x_2$ we have

$$d(x_1, x_2) = \alpha > 0. \tag{3}$$

From the triangle inequality

$$d(x_1, x_2) \leq d(x_n, x_1) + d(x_n, x_2). \tag{4}$$

From (1) and (2) we obtain

$$\lim_{n \to \infty} d(x_n, x_1) = \lim_{n \to \infty} d(x_n, x_2) = 0. \tag{5}$$

This is a contradiction with (3). Hence $x_1 = x_2$. The sequence can have only one limit.

The same result can be reached "graphically." Since $x_1 \neq x_2$

$$d(x_1, x_2) = \alpha > 0. \tag{6}$$

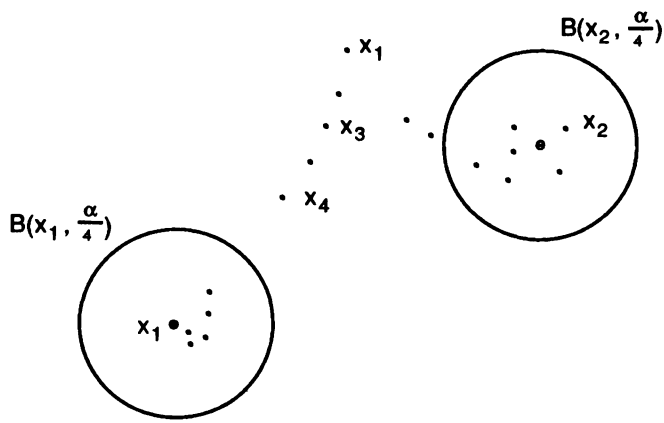

FIGURE 1

Balls $B(x_1, {}^{\alpha}/_{u})$ and $B(x_2, {}^{\alpha}/_{u})$ are disjoint and each contains all but a finite number of elements of (x_n). Contradiction.

● PROBLEM 10-19

Let X be the set of all functions from [0,1] into itself with the metric d defined as follows

$$d(f, g) = \text{least upper bound } \{ \, |f(x) - g(x)| : x \in [0, 1] \, \}. \tag{1}$$

Check the convergence of the sequence (f_n) where $f_n(x) = x^n$.

SOLUTION:

The elements of the sequence are

$$x, x^2, x^3, x^4, \ldots, x^n, \ldots \tag{2}$$

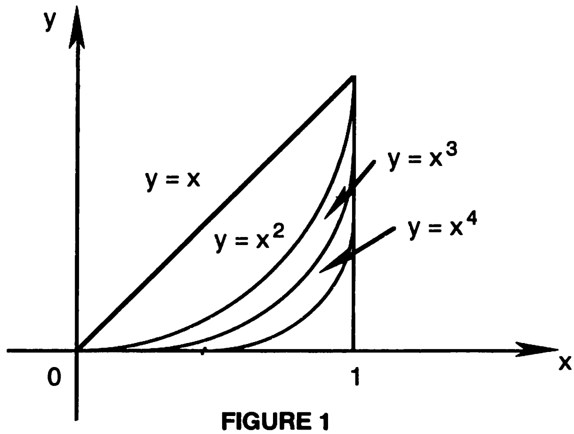

FIGURE 1

Figure 1 shows the first few elements of the sequence. We see that the sequence "converges" to the function

$$f(x) = \begin{cases} 0 & \text{for } x \neq 1 \\ 1 & \text{for } x = 1 \end{cases} \tag{3}$$

Here, the sequence of continuous functions "converges pointwise" to the function which is not continuous.

It converges to f in the sense of metric defined by (1).

$$d(f_n, f) = \text{least upper bound } \{ \, | x^n - f(x) | : x \in [0, 1] \, \} \tag{4}$$

and

$$\lim_{n \to \infty} d(f_n, f) = 0. \tag{5}$$

● PROBLEM 10-20

Prove the following:

THEOREM

A sequence of points $z_n = (x_n, y_n)$ of the space $Z = X \times Y$ is convergent to the point $z = (x, y)$, if and only if

$$x = \lim_{n \to \infty} x_n \tag{1}$$

$$y = \lim_{n \to \infty} y_n \tag{2}$$

■

SOLUTION:

Suppose $x = \lim_{n \to \infty} x_n$ and $y = \lim_{n \to \infty} y_n$.

Let $\varepsilon > 0$, then $k \in N$ exists such that for $n > k$

$$d_1(x_n, x) < \varepsilon \quad \text{and} \quad d_2(y_n, y) < \varepsilon. \tag{3}$$

Let us define

$$d(z_n, z) = \sqrt{d_1^2(x_n, x) + d_2^2(y_n, y)}. \tag{4}$$

Then

$$d(z_n, z) < \varepsilon\sqrt{2} \tag{5}$$

and

$$\lim_{n \to \infty} z_n = z. \tag{6}$$

Then k exists such that for $n > k$

$$d(z_n, z) < \varepsilon \tag{7}$$

and

$$d(z_n, z) = \sqrt{d_1^2(x_n, x) + d_2^2(y_n, y)} < \varepsilon. \tag{8}$$

Thus

$$d_1(x_n, x) < \varepsilon \quad \text{for} \quad n > k \tag{9}$$

i.e.,

$$\lim_{n \to \infty} x_n = x \tag{10}$$

Similarly, we obtain

$$d_2(y_n, y) < \varepsilon \quad \text{for} \quad n > k \tag{11}$$

i.e.,

$$\lim_{n \to \infty} y_n = y. \tag{12}$$

● PROBLEM 10-21

Show that the function $f(x) = 2x + 1$ is continuous,

$$f : R \to R \tag{1}$$

SOLUTION:

DEFINITION OF A CONTINUOUS FUNCTION

A function $f: X \to Y$, where (X, d) and (Y, d_1) are metric spaces, is said to be continuous, if for every $f(a) \in Y$ and any positive number $\varepsilon > 0$, there is a positive number $\delta > 0$ such that if $x \in B(a, \delta)$, then $f(x) \in B(f(a), \varepsilon)$.

■

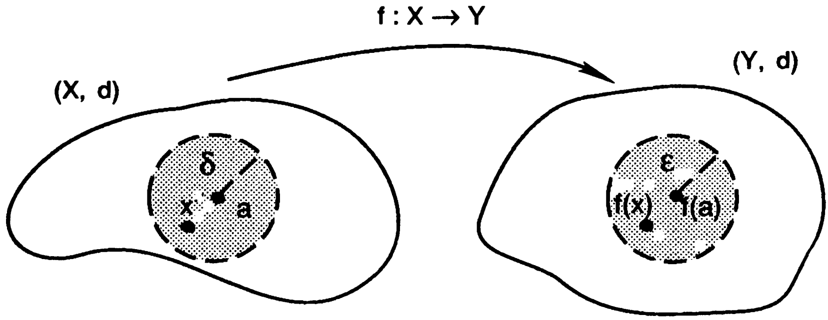

FIGURE 1

$$(x \in B(a, \delta)) \Rightarrow (f(x) \in B(f(a), \varepsilon))$$

In practical applications, this definition is rather difficult to deal with. Some other methods of establishing the continuity of functions will be shown later. Let us rewrite the definition using symbolic notation.

$f: (X, d) \to (Y, d_1)$ is continuous if

$$\forall f(a) \in Y \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that}$$

$$(d(x, a) < \delta) \Rightarrow (d_1(f(a), f(x)) < \varepsilon)$$

Let $f(a) = 2a + 1$ and $\varepsilon > 0$. We can choose $\delta = \varepsilon/2$, if $|x - a| < \delta = \varepsilon/2$, then

$$|f(x) - f(a)| = |2x + 1 - 2a - 1| =$$

$$= |2x - 2a| = 2|x - a| < 2 \cdot \varepsilon/2 = \varepsilon.$$

Hence, the function is continuous.

● PROBLEM 10-22

To determine whether or not a function is continuous, this theorem can be used:

THEOREM

$f: (X, d) \to (Y, d_1)$

$$(f \text{ is continuous}) \Leftrightarrow (\forall B \subset Y, B \text{ is open } f^{-1}(B) \text{ is open}) \qquad (1)$$

■

Prove it.

SOLUTION:

$\Rightarrow$ f is continuous and A is an open subset of Y. We need to show that

$$f^{-1}(A) = \{ x \in X : f(x) \in A \} \qquad (2)$$

is an open subset of X.

Let $x \in f^{-1}(A)$, then $f(x) \in A$. Since A is open there exists an $\varepsilon > 0$ such that

$$B(f(x), \varepsilon) \subset A. \qquad (3)$$

Since f is continuous, then there exists a $\delta > 0$ such that

$$f[B(x, \delta)] \subset B(f(x), \varepsilon) \subset A \qquad (4)$$

as shown in Figure 1.

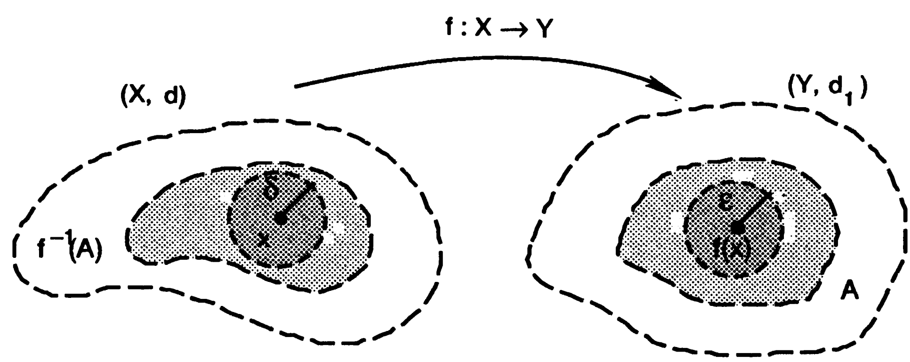

FIGURE 1

Therefore

$$B(x, \delta) \subset f^{-1}(A). \qquad (5)$$

For $x \in f^{-1}(A)$, a $\delta > 0$ exists such that (5) is true; hence $f^{-1}(A)$ is an open subset of X.

$\Leftarrow$ Suppose that if A is an open subset of Y, then $f^{-1}(A)$ is an open subset of X. If $f(a) \in Y$ and there exists an $\varepsilon > 0$, then

$$B(f(a), \varepsilon) \qquad (6)$$

is an open subset of Y.

Thus,

$$f^{-1}[B(f(a), \varepsilon)] \tag{7}$$

is an open subset of X. There exists a $\delta > 0$ such that

$$B(a, \delta) \subset f^{-1}[B(f(a), \varepsilon)] \tag{8}$$

or

$$f(B(a, \delta)) \subset B(f(a), \varepsilon). \tag{9}$$

Therefore f is continuous.

● PROBLEM 10–23

Prove the following useful:

THEOREM

Let $f : (X, d) \to (Y, d_1)$ then

$$(f \text{ is continuous}) \Leftrightarrow$$

$$(\text{for any } B(y, r) \text{ in } Y, f^{-1}(B) \text{ is an open set of } X) \tag{1}$$

■

SOLUTION:

$\Rightarrow$ Suppose f is continuous and B is any ball in Y. Then B is an open subset of Y. By theorem of Problem 10–22, $f^{-1}(A)$ is an open subset of X.

$\Leftarrow$ Suppose for any ball $B \subset Y, f^{-1}(A)$ is an open subset of X.

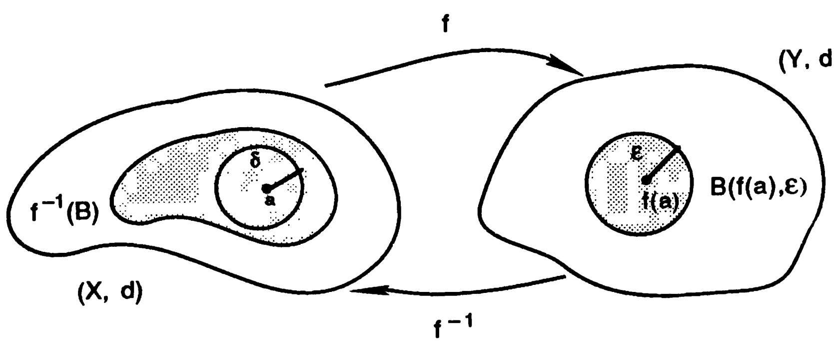

FIGURE 1

Let $f(a)$ be any point in Y and ε be any positive number. Then $B(f(a), \varepsilon)$ is a ball in Y.

The set $f^{-1}[B(f(a), \varepsilon)]$ is an open subset of X. We have

$$a \in f^{-1}[B(f(a), \varepsilon)]. \tag{2}$$

Set $f^{-1}(A)$ is an open subset of X (not necessarily a ball). Hence a ball $B(a, \delta)$ exists such that

$$B(a, \delta) \subset f^{-1}[B(f(a), \varepsilon)]. \tag{3}$$

Thus

$$(x \in B(a, \delta)) \Rightarrow (f(x) \in B(f(a), \varepsilon)).$$

Function f is continuous.

● PROBLEM 10-24

Consider the function

$$f(x, y) = x \tag{1}$$

$$f : (R^2, d_1) \rightarrow (R, d_2) \tag{2}$$

where

$$d_1((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{3}$$

$$d_2(x, y) = |x - y| \tag{4}$$

Show that f is continuous.

SOLUTION:

We shall use the theorem proved in Problem 10–23.

Let $B(a, r)$ be any ball in (R, d_2). Then $B(a, r)$ is an open interval $(a - r, a + r)$ in R.

The set $f^{-1}[B(a, r)]$ is

$$f^{-1}[(a - r, a + r)] = \{(x, y) \, ; \, a - r < x < a + r\}.$$

This set is shown in Figure 1; it is an open subset of R^2.

Hence, the function defined by (1) and (2) is continuous.

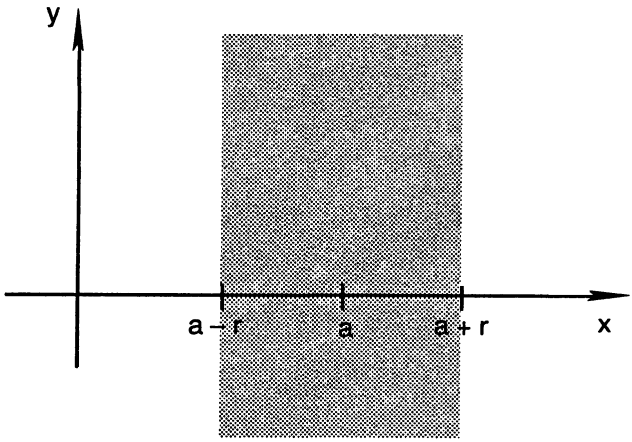

FIGURE 1

● PROBLEM 10-25

Find the limit of the sequence

$$\sin\left(1 + {}^{1}/_{n}\right)^{n} \tag{1}$$

SOLUTION:

Consider the sequence $(1 + {}^{1}/_{n})^{n}$. Its limit is

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n} = e. \tag{2}$$

Now we shall apply the following:

THEOREM

Let $f : (X, d) \to (Y, d_1)$. Function f is continuous, if and only if given any sequence (x_n), such that

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} f(x_n) = f(x). \qquad \blacksquare$$

A continuous function preserves the convergence of a sequence. Function $f(x) = \sin x$ is continuous. Therefore, since

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n} \to e \tag{3}$$

we have

$$\lim_{n\to\infty}\left\{\sin\left(1 + \frac{1}{n}\right)^{n}\right\} = e. \tag{4}$$

Sometimes we shall use

$$x_n \underset{n\to\infty}{\longrightarrow} x \tag{5}$$

to indicate

$$\lim_{n \to \infty} x_n = x.$$

● PROBLEM 10–26

1. Which of the following sets are bounded?

   closed interval $[0, 1]$ in R

   ball $B(0, 1)$ in R^2

   half-line $x < 0$ in R

2. Which of the following functions are bounded?

   $f(x) = x^2$

   $f(x) = \sin x$

SOLUTION:

1. Let A be a subset of (X, d).

DEFINITION

Consider the set A. The diameter of A, denoted by *delta* (A) is the least upper bound of the distances $d(x, y)$ between all pairs of points $x, y \in A$. Sets with finite diameter are said to be bounded. For simplicity, we write "least upper bound" as "lub."

■

Interval $[0, 1]$ is bounded in (R, d) where $d(x, y) = |x - y|$.
The ball $B(0, 1)$ is bounded.
Half-line, $x < 0$ is not bounded.

2. **DEFINITION**

A mapping $f: X \to Y$ where Y is a metric space, is called bounded if the set $f(x)$ is bounded.

■

Function

$$f: R \to R$$

$$f(x) = x^2$$

maps R onto $[0, \infty)$. Hence

$$f(R) = [0, \infty).$$

The set $[0, \infty)$ is not bounded, hence the function is not bounded.

Function $f(x) = \sin x$ maps R onto $[-1, 1]$ hence the function is bounded.

● PROBLEM 10-27

Prove the following:

THEOREM 1

The set $\Phi(X, Y)$ of all bounded mappings $f : X \to Y$, where X is an arbitrary set and (Y, d) a metric space, is a metric space with metric defined by

$$d_1(f, g) = \sup d(f(x), g(x)). \tag{1}$$

■

SOLUTION:

First let us prove:

THEOREM 2

If f and g are bounded mappings to the set X into the metric space (Y, d), then the metric

$$d_1(f, g) = \sup d(f(x), g(x)) \tag{2}$$

is finite.

■

Let a be a given element of X. Then

$$d(f(x), g(x)) \le d(f(x), f(a)) + d(f(a), g(x)) \le$$

$$\le d(f(x), f(a)) + d(f(a), g(a)) + d(g(a), g(x)). \tag{3}$$

Hence

$$d_1(f, g) \le \delta[f(X)] + d(f(a), g(a)) + \delta[g(X)]. \tag{4}$$

■

We have

$$d_1(f, g) \geq 0 \quad \text{for all } f, g \tag{5}$$

$$d_1(f, g) = 0 \quad \text{iff } f = g \tag{6}$$

Indeed, if sup $d(f(x), g(x)) = 0$ then $f(x) = g(x)$ for all $x \in X$.

$$d_1(f, g) = \mathrm{d}_1(g, f) \text{ for all } f, g \tag{7}$$

$$d_1(f, g) = \sup d(f(x), g(x)) \leq$$

$$\leq \sup [d(f(x), h(x)) + d(f(x), g(x)] \leq$$

$$\leq \sup d(f(x), h(x)) + \sup d(h(x), g(x)) =$$

$$= d_1(f, h) + d_1(h, g). \tag{8}$$

Hence d_1 defined in (1) is a metric.

● PROBLEM 10-28

1. Show that if $A \neq \phi$ and $B \neq \phi$ and $A \subset B$, then

$$\delta(A) \leq \delta(B). \tag{1}$$

2. Show that if $A \cap B \neq \phi$, then

$$\delta(A \cup B) \leq \delta(A) + \delta(B). \tag{2}$$

SOLUTION:

1. We have

$$\delta(A) = \text{lub } d(x, y) \tag{3}$$

where $x, y \in A$.

Note that since $A \subset B$

$$d(x, y) \leq d(x, y) \tag{4}$$

for $x, y \in A$.

Then,

$$\underset{\text{where } x, y \in A}{\text{lub } d(x, y)} \leq \underset{\text{where } x, y \in B}{\text{lub } d(x, y)} \tag{5}$$

and

$$\delta(A) \leq \delta(B). \tag{6}$$

2.

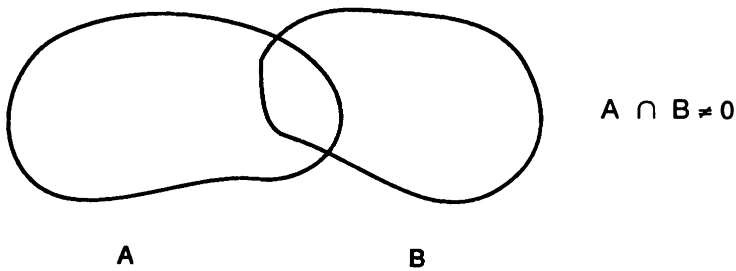

FIGURE 1

Let us define the distance between sets A and B

$$\delta(A, B) = \text{greatest lower bound } \{d(a, b) : a \in A, b \in B\}. \tag{7}$$

Then, for any subsets A and B of (X, d)

$$\delta(A \cup B) \leq \delta(A) + \delta(B) + \delta(A, B). \tag{8}$$

But $A \cap B \neq 0$ therefore

$$\delta(A, B) = 0. \tag{9}$$

Hence

$$\delta(A \cup B) \leq \delta(A) + \delta(B). \tag{10}$$

● PROBLEM 10-29

Let (X, d) be a metric space. Show that a subset A of X is closed, if and only if for any $x \in X - A$,

$$\delta(x, A) \neq 0 \tag{1}$$

where

$$\delta(x, A) =: \text{greatest lower bound or (glb) } \{d(x, a) : a \in A\}. \tag{2}$$

SOLUTION:

Suppose A is a closed set, then $X - A$ is open. Hence for any $x \in X - A$, a ball $B(x, r)$ exists such that

$$B(x, r) \subset X - A \tag{3}$$

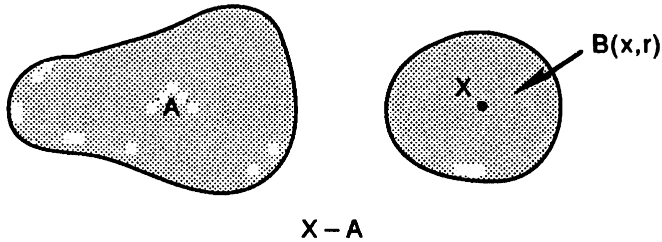

FIGURE 1

Then

$$\delta(x, A) \geq r \tag{4}$$

and

$$\delta(x, A) \neq 0. \tag{5}$$

Now suppose, for any $x \in X-A$, $\delta(x, A) \neq 0$.
Let

$$r = \delta(x, A) \tag{6}$$

then

$$B(x, r) \subset X-A. \tag{7}$$

Hence $X-A$ is open and A is closed

$$A = X-(X-A). \tag{8}$$

● PROBLEM 10–30

Let (X, d) be a metric space. Show that if A and A' are closed subsets of X, such that $A \cap A' = \phi$, then open sets B and B' exist such that

$$A \subset B, \quad A' \subset B', \quad B \cap B' = \phi. \tag{1}$$

SOLUTION:

Let us define for each $x \in A$

$$r_x = {}^1\!/_2\, \delta(x, A') \tag{2}$$

and for each $x' \in A'$

$$r'_{x'} = {}^1\!/_2\, \delta(x', A) \tag{3}$$

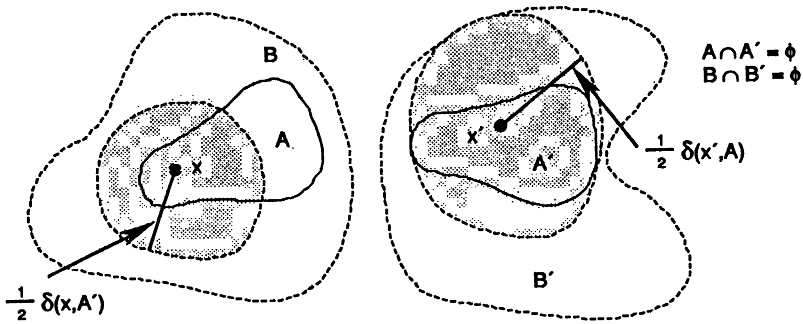

FIGURE 1

We define

$$B = \bigcup_{A} B(x, r_x)$$

$$B' = \bigcup_{A'} B(x', r'_{x'}) \tag{4}$$

Both sets B and B' are open because they are the union of a family of open sets (balls).

Since the union has taken over all the elements of A

$$A \subset B \tag{5}$$

similarly

$$A' \subset B'. \tag{6}$$

We will show that $B \cap B' = \phi$.

Suppose on the contrary

$$x \in B \cap B'. \tag{7}$$

Then for some $y \in A$

$$\delta(y, x) < r_y \tag{8}$$

and for some $y' \in A'$

$$\delta(y', x) < r'_{y'} \tag{9}$$

Suppose $r_y \geq r'_{y'}$, then

$$\delta(y, y') \leq \delta(y, x) + \delta(y', x) < r_y + r'_{y'} \leq$$

$$\leq 2r_y = \delta(y, A') \tag{10}$$

where $\delta(y, y') = d(y, y')$. But

$$\delta(y, y') \geq d(y, A'). \tag{11}$$

Hence a contradiction,

$$B \cap B' = \phi. \tag{12}$$

Two closed disjoint sets are well separable, in the sense that it is possible to put them into two open sets, which again are disjoint.

● PROBLEM 10–31

Consider the metric space (R^2, d) where d is the Pythagorean metric. Let A be the hyperbola $y = {}^1/_x$ and A' the x -axis. Find the open sets B and B' as described in Problem 10–30.

SOLUTION:

Both sets

$$A = \{ (x, y) : y = {}^1/_x, x \neq 0\} \tag{1}$$

and

$$A' = \{ (x, y) : y = 0\} \tag{2}$$

are closed and

$$A \cap A' = \phi. \tag{3}$$

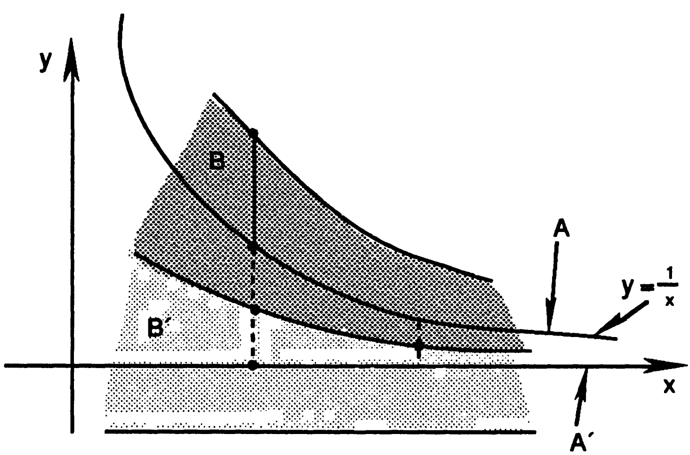

FIGURE 1

At first glance, it seems impossible to find the separating open sets B and B' because

$$\delta(A, A') = 0 \tag{4}$$

i.e., the distance between A and A' is zero. Function $y = {}^1/_x$ has the x-axis for an asymptote. Such sets B and B', however, exist.

$$B = \{ (x, y) : {}^1/_{2x} < y < {}^1/_x + 1\} \tag{5}$$

$$B' = \{ (x, y) : -1 < y < {}^1/_{2x} \}. \tag{6}$$

Both B and B' are open. We have

$$A \subset B \text{ and } A' \subset B' \tag{7}$$

$B \cap B' = \phi$.

● PROBLEM 10-32

> 1. Let (R, d) be the space of real numbers with the absolute value metric and let
>
> $$A = \{{}^1/_n : n \in N\}. \tag{1}$$
>
> Find the closure of A.
>
> 2. Show that for any subset A of X, the closure of A is a closed subset of X.

SOLUTION:

DEFINITION OF THE CLOSURE OF A

Let (X, d) be the metric space and $A \subset X$. The closure of A, denoted by $\overline{A}$ (sometimes by $C(A)$, is defined by

$$\overline{A} =: \{x \in X : \delta(x, A) = 0\}. \tag{2}$$

■

1. Since ${}^1/_n \to 0$, we have

$$\delta(0, A) = 0. \tag{3}$$

For each $x \in A$,

$$\delta(x, A) = 0 \tag{4}$$

Therefore

$$A \subset \overline{A}. \tag{5}$$

From (3) and (5) we obtain

$$A \cup \{0\} \subset \overline{A}. \tag{6}$$

Suppose $x \in \overline{A}$ and $x \notin A \cup \{0\}$. Then

$$\delta(x, A) > 0. \tag{7}$$

We conclude that

$$A \cup \{0\} = \overline{A}. \tag{8}$$

2. Suppose $\overline{A}$ is not closed. Then $X - \overline{A}$ is not open and there is an element $x \in X - \overline{A}$ such that for every $\varepsilon > 0$

$$B(x, \varepsilon) \cap A \neq \phi. \tag{9}$$

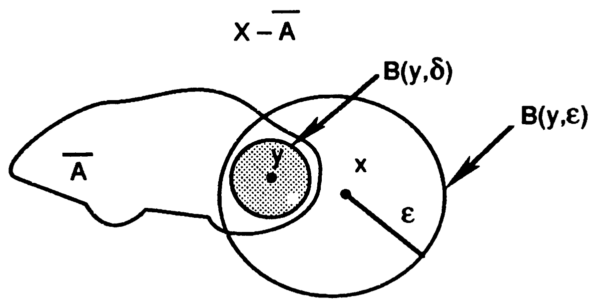

FIGURE 1

Choose $$y \in \overline{A} \cap B(x, \varepsilon). \tag{10}$$

Since $B(x, \varepsilon)$ is open, $\delta > 0$ exists such that

$$b(y, \delta) \text{ A } B(x, \varepsilon). \tag{11}$$

On the other hand $y \in \overline{A}$, then

$$\delta(y, A) = 0. \tag{12}$$

Therefore, a exists such that

$$a \in A \cap B(y, \delta) \subset B\,(\mathrm{x}, \varepsilon). \tag{13}$$

Hence, for any $\varepsilon > 0$, there is $a \in A$ such that $\delta(x, a) < \varepsilon$

$$\text{greatest lower bound } \{\ \delta(x, a) : a \in A\} = \delta(x, A) = 0. \tag{14}$$

Therefore $x \in \overline{A}$, but we assumed that $x \in X - \overline{A}$. Contradiction.

Prove:
THEOREM

The limit of a uniformly convergent sequence of bounded mappings is bounded.

■

SOLUTION:

DEFINITION OF UNIFORM CONVERGENCE

Let $f_n : X \to Y$, $n \in N$, where X is an arbitrary set and (Y, d) is a metric space. The sequence (f_n) is said to converge uniformly to f of

$$\forall \varepsilon > 0 \quad \exists k \in N \quad \forall x \in X \quad \forall n \geq k \quad d(f_n(x), f(x)) < \varepsilon. \tag{1}$$

■

Consider the space $\Phi(X, Y)$ of all bounded mappings $f : X \to Y$ with the metric defined by

$$d(f, g) =: \sup \{d(f(x), g(x)) : x \in X\}. \tag{2}$$

We have

$$\left(\lim_{n\to\infty} f_n = f\right) \equiv \left(\lim_{n\to\infty} d(f_n, f) = 0\right) \equiv$$
$$\equiv \forall \varepsilon > 0\ \exists k \in N\ \forall n \geq k \sup\{d(f_n(x), f(x)\} < \varepsilon \equiv$$
$$\equiv \forall \varepsilon > 0\ \exists k \in N\ \forall n \geq k \forall x \in X\ d(f_n(x), f(x)) < \varepsilon. \tag{3}$$

Thus, in the space $\Phi(X, Y)$ the condition $\lim f_n = f$ means that the sequence (f_n) converges uniformly to f.

Let $\varepsilon > 0$ and let $f_n \to f$, (f_n) be a uniformly convergent sequence of bounded mappings. We choose n such that

$$d(f_n, f) < \varepsilon. \tag{4}$$

We have

$$d(f(x_1), f(x_2)) \leq d(f(x_1), f_n(x_1)) + d(f_n(x_2), f(x_2)) +$$
$$+ d(f_n(x_1), f_n(x_2)). \tag{5}$$

Thus

$$\delta[f(X)] \leq \delta[f_n(X)] + 2\varepsilon \tag{6}$$

and f is bounded.

CHAPTER 11

TOPOLOGICAL SPACES

1. Let X denote any set and $P(X)$ a family of all subsets of X. Show that $P(X)$ is a topology.

2. Find a topology on X, which contains as few as possible subsets of X.

SOLUTION:

DEFINITION OF TOPOLOGY

T is a topology on X, if and only if

1. $T \subset P(X)$

2. $\phi \in T$ and $X \in T$

3. If $A_1, A_2 \in T$, then $A_1 \cap A_2 \in T$, i.e., each finite intersection of members of T is also a member of T.

4. Each union of members of T is also a member of T.

■

DEFINITION OF TOPOLOGICAL SPACE

A couple (X, T) consisting of a set X and a topology T on X is called a topological space.

■

1. Since $P(X)$ is a family of all subsets of X, $P(X)$ is a topology. This topology is called the discrete topology on X. It contains the maximum possible number of sets.

2. Let X represent any set. The collection $T = \{\phi, X\}$ is a topology on X.
This topology is called the indiscrete topology (or trivial topology) on X. It contains the fewest possible number of sets.

Elements of topological spaces are called points. The members of T are called the open sets of the topological space (X, T).

Let $X = \{a, b, c, d\}$ and $T_0 = \{\{a\}, \{a, b\}, \{a, c, d\}\}$. Find a few possible topologies T on X, such that

$$T_0 \subset T. \tag{1}$$

SOLUTION:

Each union of members of T is also a number of T (see Problem 11–1). Hence

$$\{a, b\} \cup \{a, c, d\} = \{a, b, c, d\} = X. \tag{2}$$

One of the possible topologies on X is

$$T_1 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}\}$$

$$T_0 \subset T_1. \tag{3}$$

Suppose $\{b\} \in T_2$, then

$$T_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}\}. \tag{4}$$

T_2 is a topology and $T_0 \subset T_2$.

Suppose $\{c\} \in T_3$, then

$$T_3 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{c\}, \{a, c\}, \{a, b, c\}\}. \tag{5}$$

Suppose $\{d\} \in T_4$ then

$$T_4 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{d\}, \{a, d\}, \{a, b, d\}\}. \tag{6}$$

Suppose $\{b\} \in T_5$ and $\{c\} \in T_5$ then

$$T_5 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{b\}, \{c\}, \{a, c\},$$

$$\{a, b, c\}, \{b\, c\}\}. \tag{7}$$

Suppose $\{b\} \in T_6$ and $\{d\} \in T_6$ then

$$T_6 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{b\}, \{d\}, \{a, d\},$$

$$\{b, d\}, \{a, b, d\}\}. \tag{8}$$

Suppose $\{c\} \in T_7$ and $\{d\} \in T_7$ then

$$T_7 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{c\}, \{d\}, \{a, c\},$$

$$\{a, d\}, \{a, b, c\}, \{a, b, d\}, \{c, d\}\}. \tag{9}$$

Now let $T_8 = P(X)$

$$T_8 = \{\phi, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\},$$

$$\{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}. \tag{10}$$

Suppose $\{a, c\} \in T_9$ then

$$T_9 = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{a, c, d\}, \{a, b, c\}\}. \tag{11}$$

● PROBLEM 11-3

Let R represent the set of real numbers. Subset $A \subset R$ is called open if for each $x \in A$, there is an $r > 0$, such that

$$B(x, r) =: \{ y : |y - x| < r \} \subset A \tag{1}$$

(R, d), where $d(x, y) = |x - y|$ is a metric space. Let T denote the family of all open sets (in the sense defined above). Verify that T is a topology on R.

SOLUTION:

Obviously ϕ and R are open sets.

$$\phi \in T, \quad R \in T. \tag{2}$$

Suppose $A_1, \ldots, A_n$ are open sets, then

$$\bigcap_1^n A_k \in T.$$

Indeed

$$(x \in \bigcap_1^n A_k) \Rightarrow (\forall k, x \in A_k) \Rightarrow$$

$$\Rightarrow (\forall k \exists r_k > 0 : B(x, r_k) \subset A_k) \Rightarrow$$

$$\Rightarrow \left(\begin{array}{c} B(x,r) \subset \bigcap_1^n A_k \\ \text{where } r = \min\{r_1, r_2, \ldots, r_n\} \end{array} \right) \tag{3}$$

Suppose each member of $\{A_\omega : \omega \in \Omega\}$ is open, then $\bigcup_\omega A_\omega$ is also open.

$$(x \in \bigcup_\omega A_\omega) \Rightarrow (\exists w : x \in A_\omega) \Rightarrow$$

$$\Rightarrow (\exists r > 0 : B(x, r) \subset A_w \subset \bigcup_\omega A_\omega) \tag{4}$$

Topology defined above is called the Euclidean topology on R'. The topological space (R', T) is called the Euclidean 1-space.

● PROBLEM 11-4

Let (X, d) denote a metric space. Consider the family T of all d-open subsets of X, i.e., subsets which are open in the sense of metric d.

Show that T is a topology on (X, d).

SOLUTION:

Let us recall the definition of an open set A. A is open iff

$$\forall x \in A \quad \exists B(x, r) : B(x, r) \subset A. \tag{1}$$

An empty set and the whole space are open sets

$$\phi \in T, \quad X \in T \tag{2}$$

Suppose $A_1, \ldots, A_n$ are open sets. Then $\bigcap_1^n A_k$ is an open set.

$$\begin{gathered}
(x \in \bigcap_1^n A_k) \Rightarrow (\forall k : x \in A_k) \Rightarrow \\
\Rightarrow (\forall k \exists r_k : B(x, r_k) \subset A_k) \Rightarrow \\
\Rightarrow \left(\forall k : B(x,r) \subset A_k \text{ where } r = \min\{r_1, \ldots, r_n\}\right) \Rightarrow \\
\Rightarrow (B(x,r) \subset \bigcap_1^n A_k).
\end{gathered} \tag{3}$$

Now, let A_α denote a family of open sets, then $\bigcup_\alpha A_\alpha$ is an open set.

$$\begin{gathered}
(x \in \bigcup_\alpha A_\alpha) \Rightarrow (\exists \alpha : x \in A_\alpha) \Rightarrow \\
\Rightarrow (\exists B(x,r) : B(x,r) \subset A_\alpha \subset \bigcup_\alpha A_\alpha.
\end{gathered} \tag{4}$$

Topology T described here for the metric space (X, d) is called the metric topology induced on X by d. Each metric space (X, d) can become an "automatically" topological space (X, T). Different metrics on X will induce in general different topologies on X.

● PROBLEM 11-5

1. Define the topological space Z and the Sierpinski space.

2. Suppose T is a topology on X, consisting of

$$T = \{\phi, X, A, B\} \tag{1}$$

where A and B are non-empty distinct proper subsets of X. Determine conditions on A and B.

SOLUTION:

1. Let $X = \{0, 1\}$. Then the discrete topology on X, $T = P(X)$, is denoted by Q.

$$T = \{\phi, \{0, 1\}, \{0\}, \{1\}\} \tag{2}$$

The set $\{0, 1\}$ with topology

$$T = \{\phi, \{0, 1\}, \{0\}\} \tag{3}$$

is called the Sierpinski space.

2. Since T defined by (1) is a topology

$$A \cap B \in T = \{\phi, X, A, B\}. \tag{4}$$

We have either

$$A \cap B = \phi \tag{5}$$

or

$$A \cap B = A \tag{6}$$

or

$$A \cap B = B. \tag{7}$$

Suppose $A \cap B = \phi$. Then

$$A \cup B \neq A \quad \text{and} \quad A \cup B \neq B \quad \text{and} \quad A \cup B \neq \phi.$$

Hence

$$A \cup B = X \tag{8}$$

$\{A, B\}$ is a partition of X.

Suppose now $A \cap B = A$ then

$$\phi \subset A \subset B \subset X. \tag{9}$$

If $A \cap B = B$ then

$$\phi \subset B \subset A \subset X. \tag{10}$$

Let $f : X \to Y$ denote a function of $X \neq \phi$ onto a topological space (Y, T_0). Show that the family T of subsets of X, defined by

$$T = \{f^{-1}(B) : B \in T_0\} \tag{1}$$

is a topology on X.

SOLUTION:

T_0 is a topology on Y. Therefore

$$\phi, Y \in T_0 \tag{2}$$

Since f is onto

$$f^{-1}(Y) = X. \tag{3}$$

Also
$$f^{-1}(\phi) = \phi. \tag{4}$$

Thus
$$\phi, X \in T. \tag{5}$$

Let $\{A_\omega : \omega \in \Omega\}$ represent a family of sets in T. Then for each ω

$$f^{-1}(B_\omega) = A_\omega \quad B_\omega \in T_0 \tag{6}$$

We have

$$\bigcup_\omega A_\omega = \bigcup_\omega f^{-1}(B_\omega) = f^{-1}(\bigcup_\omega B_\omega) \tag{7}$$

Since T_0 is a topology

$$\bigcup_\omega B_\omega \in T_0 \tag{8}$$

therefore $\bigcup_\omega A_\omega \in T$.

Let $A_1, A_2 \in T$. Then sets $B_1, B_2, \in T_0$ exist, such that

$$f^{-1}(B_1) = A_1 \quad \text{and} \quad f^{-1}(B_2) = A_2 \tag{9}$$

Then

$$A_1 \cap A_2 = f^{-1}(B_1) \cap f^{-1}(B_2) = f^{-1}(B_1 \cap B_2). \tag{10}$$

Since T_0 is a topology and $B_1, B_2 \in T_0$, the intersection $B_1 \cap B_2 \in T_0$. Therefore

$$f^{-1}(B_1 \cap B_2) \in T \quad \text{and} \quad A_1 \cap A_2 \in T. \tag{11}$$

Thus (X, T) is a metric space.

PROBLEM 11-7

Consider set N with the family T of its subsets consisting of ϕ and all sets of the form

$$A_k = \{k, k+1, k+2, \ldots\} \quad k = 1, 2, 3, \ldots . \tag{1}$$

Show that T is a topology on N.

SOLUTION:

$$A_1 = \{1, 2, 3, \ldots\} \tag{2}$$

Thus

$$\phi \in T \quad \text{and} \quad N \in T \tag{3}$$

Suppose $A_k, A_l \in T$. Then either $k > l$ or $k < l$ or $k = l$. If $k > l$, then

$$A_k \cap A_l = A_k \in T \tag{4}$$

if $k < l$, then

$$A_k \cap A_l = A_l \in T. \tag{5}$$

Also if $k = l$, then $A_k \cap A_k = A_k \in T$. Thus $A_k \cap A_l = T$.

Let $\{A_i\}$ represent a family of sets of the form (1). Then

$$p = \min \{i_1, i_2, \ldots\} \tag{6}$$

and

$$\bigcup_i A_i = A_p = \{p, p+1, p+2, \ldots\}. \tag{7}$$

Thus T is a topology on N.

PROBLEM 11-8

1. Show that intersection of any family of topologies on X is also a topology on X.

2. Show that the union of topologies does not have to be a topology.

SOLUTION:

1. Let $\{T_\omega : \omega \in \Omega\}$ be a family of topologies on X.

$$\forall\, \omega \in \Omega : \phi, X \in T_\omega$$

Hence $$\phi, x \in \bigcap_{\omega} T_{\omega}. \tag{1}$$

Suppose $A, B \in \bigcap_{\omega} T_{\omega}$.

$$(A, B \in \bigcap_{\omega} T_{\omega}) \Rightarrow (\forall\, \omega : A, B \in T_{\omega}) \Rightarrow$$

$$\Rightarrow (\forall\, \omega : A \cap B \in T_{\omega}) \Rightarrow (A \cap B \in \bigcap_{\omega} T_{\omega}). \tag{2}$$

Suppose $\{A_i\}$ is a family of sets such that

$$(\forall\, i : A_i \in \bigcap_{\omega} T_{\omega}) \Rightarrow (\forall\, i\, \forall\, \omega\; A_i \in T_{\omega}) \Rightarrow$$

$$\Rightarrow (\forall\, \omega : \bigcup_i A_i \in T_{\omega}) \Rightarrow (\bigcup_i A_i \in \bigcap_{\omega} T_{\omega}). \tag{3}$$

Hence if $\{T_{\omega} : \omega \in \Omega\}$ is a family of topologies on X then $\bigcap_{\omega} T_{\omega}$ is a topology on X.

2. Consider the set

$$X = \{a, b, c\} \tag{4}$$

and two topologies on X

$$T_1 = \{\phi, X, \{a\}\} \tag{5}$$

and

$$T_2 = \{\phi, X, \{b\}\} \tag{6}$$

Then

$$T_1 \cup T_2 = \{\phi, X, \{a\}, \{b\}\} \tag{7}$$

is not a topology because

$$\{a\} \cup \{b\} = \{a, b\} \notin T_1 \cup T_2. \tag{8}$$

• PROBLEM 11-9

How many distinct topologies can a set consisting of three elements have? What is their partial ordering?

SOLUTION:

Let

$$X = \{a, b, c\}. \tag{1}$$

The possible topologies on X are

$$T_1 = \{\phi, X\}$$

$$T_2 = \{\phi, X, \{a\}\}, \qquad T_3 = \{\phi, X, \{b\}\},$$

$$T_4 = \{\phi, X, \{c\}\}, \qquad T_5 = \{\phi, X, \{a, b\}\}$$

$$T_6 = \dots \tag{2}$$

and so on.

Consider the set $P(X)$

$$P(X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\} \tag{3}$$

1 topology $\{\phi, X\}$
3 topologies of the type $\{\phi, X, \{\cdot\}\}$
where $\{\cdot\}$ indicates the set consisting of one element
3 topologies of the type $\{\phi, X, \{\cdot,\cdot\}\}$
9 topologies of the type $\{\phi, X, \{\cdot\}, \{\cdot,\cdot\}\}$
3 topologies of the type $\{\phi, X, \{\cdot\}, \{\cdot,\cdot\}, \{\cdot,\cdot\}\}$
3 topologies of the type $\{\phi, X, \{\cdot\}, \{\cdot\}, \{\cdot,\cdot\}\}$
6 topologies of the type $\{\phi, X, \{\cdot\}, \{\cdot\}, \{\cdot,\cdot\}, \{\cdot,\cdot\}\}$
1 topology $P(X)$

The numbers must be added up in order to obtain the total number of topologies on $X = \{a, b, c\}$.

We can introduce the relation of inclusion among topologies. For example,

$$\{\phi, X, \{a\}\} \subset \{\phi, X, \{a\}, \{b\}, \{a, b\}\}. \tag{4}$$

This relation defines a partial order. Indeed, Let T_1, T_2, T_3 represent topologies on X, then

$$(T_1 \subset T_2) \wedge (T_2 \subset T_3) \Rightarrow (T_1 \subset T_3)$$

$$\forall T : T \subset T$$

$$(T_1 \subset T_2) \wedge (T_2 \subset T_1) \Rightarrow (T_1 = T_2). \tag{5}$$

● PROBLEM 11-10

Prove that T is the discrete topology on X if and only if every point is an open set.

SOLUTION:

Let $T = P(X)$, then T is called the discrete topology on X. Every set is an open set. We shall prove that

$$(T \text{ is the discrete topology}) \Leftrightarrow (\text{every point is an open set}). \tag{1}$$

$\Rightarrow$ Since T consists of all subsets of X

$$\forall\, x \in X : \{x\} \in T. \tag{2}$$

$\Leftarrow$ Suppose T is a topology on X, such that

$$\forall\, x \in X : \{x\} \in T. \tag{3}$$

Let A denote any subset of X. Then

$$A = \bigcup_{a \in A} \{a\}. \tag{4}$$

But all $\{a\}$ are open sets, $\{a\} \in T$ and since T is a topology, the union of open sets is an open set.

Hence, $A \in T$, we conclude, that $T = P(X)$.

• PROBLEM 11–11

Let X represent any set and S a family of its subsets, such that

1. $X, \phi \in S$.
2. The union of any two members of S is a member of S.
3. The intersection of any family of members of S is a member of S.

Let T denote a family of subsets of X, such that

$$A \in T \quad \text{iff} \quad X - A \in S. \tag{1}$$

Show that T is a topology on X.

SOLUTION:

Since $X, \phi \in S$ and $X - X = \phi, X - \phi = X$ $X, \phi \in T$.

Suppose $A_1, A_2 \in T$, then $X - A_1 \in S$ and $X - A_2 \in S$

$$(X - A_1) \cup (X - A_2) = X - (A_1 \cap A_2) \in S. \tag{2}$$

Therefore $A_1 \cap A_2 \in T$.

Suppose $\{A_\alpha\}$ is a family of subsets of T. Then $\{X - A_\alpha\}$ is a family of subsets of S and

$$\bigcap_\alpha (X - A_\alpha) \in S. \tag{3}$$

But

$$\bigcap_\alpha (X - A_\alpha) = X - \bigcup_\alpha A_\alpha. \tag{4}$$

Hence $X - \bigcup_\alpha A_\alpha \in S$ and $\bigcup_\alpha A_\alpha \in T$.

Thus T is a topology on X. Elements of T are called open sets.

DEFINITION OF A CLOSED SET

Subset B of (X, T) is closed, if $X - B$ is an open set, that is, if $X - B \in T$.

● PROBLEM 11-12

Give an example of a basis for the Euclidean topological space (R^n, T).

SOLUTION:

DEFINITION OF A BASIS

Let (X, T) denote a topological space. A family $B \subset T$ is called a basis for T if each open set (i.e., an element of T) is the union of members of B.

■

Note that if T is a topology on X, then T is a basis for T. In (R^n, T), the topology is the family of sets open in the sense of the Euclidean metric.

We shall show that in R^n

$$B = \{B(x, r) : x \in R^n, r > 0\} \tag{1}$$

is a basis for the Euclidean topology.

Let $A \in T$ denote an open subset of R^n. Then

$$A = \bigcup_{x \in A} B(x, r). \tag{2}$$

Each open set can be represented as a union of balls. Note that (1) is not the only possible basis for the Euclidean topology in R^n.

● PROBLEM 11-13

Let $B \subset T$, where T is a topology on X. Show that the following properties of B are equivalent:

1. B is a basis for T.

2. For each $A \in T$ and each $x \in A$, there is $B_\omega \in B$, such that
$$x \in B_\omega \subset A.$$

SOLUTION:

(1) $\Rightarrow$ (2) Let $A \in T$. Since B is a basis for T

$$A = \bigcup_\omega B_\omega \tag{1}$$

where $B = \{B_\omega : \omega \in \Omega\}$. $x \in A$, therefore at least one B_ω exists, such that

$$x \in B_\omega \subset A. \tag{2}$$

(1) $\Leftarrow$ (2) Suppose $A \in T$. Since for each $x \in A$, set B_ω exists, such that

$$B_\omega \in B \tag{3}$$

and

$$x \in B_\omega \subset A \tag{4}$$

we have

$$A = \bigcup_{x \in A} B_\omega, \text{ where } x \in B_\omega. \tag{5}$$

Thus B is a basis for T.

The following theorem is useful in describing open sets.

THEOREM

Let $B \subset T$ represent a basis for T. Then set A is open, i.e., $A \in T$, if and only if, for each $x \in A$, there is a $B_\omega \in B$, such that $x \in B_\omega \subset A$.

● PROBLEM 11-14

Show that R^n with the Euclidean topology has a countable basis.

SOLUTION:

Consider the family of all open balls $B_\omega(x, r)$, such that r is a rational number and all coordinates of $x = (x_1, x_2, \ldots, x_n)$ are rational numbers. This family forms a countable set.

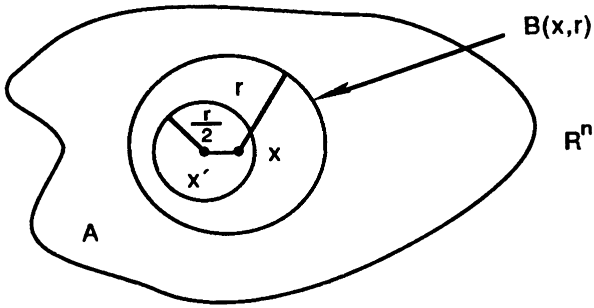

FIGURE 1

Let A denote any open subset of R^n and let $x \in A$. There is a ball $B(x, r)$, such that

$$x \in B(x, r). \tag{1}$$

We can assume that $r > 0$ is a rational number. We can always find a point x', such that

$$x' = (x'_1, x'_2, \ldots, x'_n)$$

where all coordinates of x' are rational and

$$d(x, x') < {}^r/_3. \tag{2}$$

Then

$$x \in B(x', {}^r/_2) \subset B(x, r). \tag{3}$$

Thus

$$B = \{B_\omega(x, r) : \text{all coordinates of } x \text{ are rational, and } r > 0 \text{ is rational}\} \tag{4}$$

is a countable basis of R^n.

Each open set in R^n is the union of, at the most, countably many balls.

If T is the Euclidean topology of R^n, then

$$\text{card } T = 2^{\text{card } N}. \tag{5}$$

● PROBLEM 11-15

Let B represent a basis for topology T and let B' represent a family of open sets, such that

$$B \subset B' \subset T. \tag{1}$$

Show that B' is also a basis for T.

SOLUTION:

Suppose $A \in T$ is an open subset of X.

Since B is a base for T

$$A = \bigcup_{\omega} B_{\omega} \tag{2}$$

where $B_{\omega} \in B$. Since $B \subset B'$, each $B_{\omega} \in B$ also belongs to B', $B_{\omega} \in B'$.

Therefore

$$A = \bigcup_{\omega} B_{\omega} \text{ where } B_{\omega} \in B'. \tag{3}$$

Hence B' is a base for T.

● PROBLEM 11-16

A set X and a family of all open subsets of X — so called topology T — are given. This pair (X, T) is the topological space. Sometimes we are given the basis B for topology T, then taking all unions of elements of B, we find T.

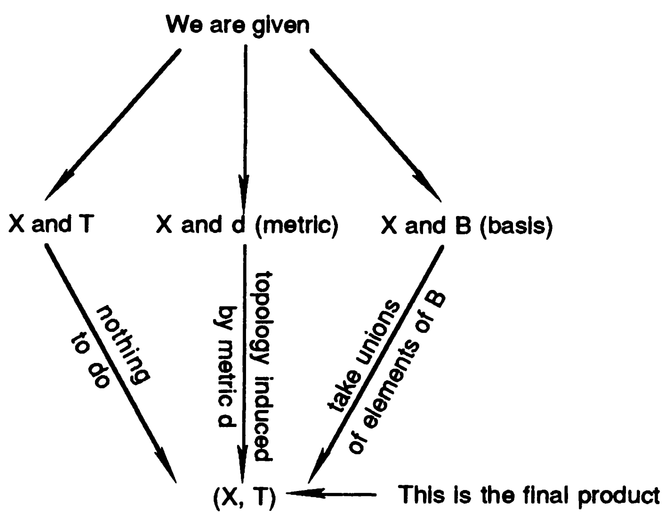

FIGURE 1

Consider this situation. A set X and a family B of its subsets are given. The question is: is this family a basis for any topology T on X? Prove the

following theorem which answers this question:

THEOREM

Let B denote a family of subsets of X, such that

1. X is the union of members of B.

2. The intersection of any two members of B is the union of members of B.

Then B is a basis for topology T on X defined by

$$T = \{A \subset X : A \text{ is the union of members of } B\}. \tag{1}$$

SOLUTION:

1. $X \in T$ because X is the union of members of B. Also $\phi \in T$, as the union of the empty subfamily of B.

2. Let $\{A_\alpha\}$ denote a family of members of T, i.e., for each α, $A_\alpha \in T$. But each A_α is the union of members of B

$$A_\alpha = \bigcup_{\beta(\alpha)} B_{\beta(\alpha)} \tag{2}$$

where $B_{\beta(\alpha)} \in B$. Then

$$\bigcup_\alpha A_\alpha = \bigcup_\alpha \bigcup_{\beta(\alpha)} B_{\beta(\alpha)} \in T. \tag{3}$$

3. Suppose A_1 and $A_2 \in T$. Then

$$A_1 = \bigcup_\alpha B_\alpha \tag{4}$$

$$A_2 = \bigcup_\beta B_\beta \tag{5}$$

where $B_\alpha, B_\beta \in B$ and

$$A_1 \cap A_2 = \bigcup_{\alpha,\beta} (B_\alpha \cap B_\beta). \tag{6}$$

The intersection of any two members of B is the union of members of B. Hence, for each α and β

$$B_\alpha \cap B_\beta = \bigcup_\gamma B_\gamma \tag{7}$$

and

$$A_1 \cap A_2 = \bigcup_{\alpha,\beta} \bigcup_\gamma B_{\gamma(\alpha,\beta)} \in T \tag{8}$$

where each γ depends on α and β. Therefore, T defined by (1) is a topology and B described in the theorem is a basis for T.

Let R represent the set of real numbers.

1. Show that the collection of open intervals is the basis for a topology T on R.

2. (R, d) where $d(x, y) = | x - y |$ is a metric space and hence a topological space (R, T') with topology induced by the metric.

Show that both topologies T and T' are the same.

SOLUTION:

1. R is the union of open intervals. Any intersection of two open intervals is either empty or again an open interval. By virtue of theorem of Problem 11–16, the collection of open intervals in R forms a basis for a topology T on R.

2. Let

$$T = \{A : A \text{ is the union of open intervals}\} \tag{1}$$

$$T' = \{B : B \text{ is open in sense of the absolute value metric}\} \tag{2}$$

We will show that

$$(A \in T) \Rightarrow (A \in T'). \tag{3}$$

If $A \in T$, then A is the union of open interals. Each open interval is a set which is open in the sense of the absolute value metric. A union of open sets is again an open set. Hence, $A \in T'$

$$(A \in T') \Rightarrow (A \in T). \tag{4}$$

Suppose A is open in the sense of the absolute value metric. Then

$$\forall\, x \in A \quad \exists\, B(x, r) : B(x, r) \subset A \tag{5}$$

But

$$B(x, r) = (x - r, x + r). \tag{6}$$

Hence

$$A = \bigcup_{x \in A} B(x, r) = \bigcup_{x \in A} (x - r, x + r). \tag{7}$$

A is the union of open intervals, $A \in T$. We conclude

$$T = T'. \tag{8}$$

Let A represent a family of subsets of X, $A \subset P(X)$. In general, there are many topologies T on X which contain A, for example $A \subset T = P(X)$. We will show that among these topologies a unique, smallest topology $T(A) \supset A$ exists.

THEOREM

Let $A = \{A_\alpha\}$ be a family of subsets of X. Then $T(A)$ is a unique, smallest topology containing A, when

$$T(A) = \{\phi, X, \text{all finite intersections of } A_\alpha, \text{all unions of finite intersections of } A_\alpha\} \qquad (1)$$

$T(A)$ is said to be generated by A, and A is a subbasis for $T(A)$.

■

Prove this theorem.

SOLUTION:

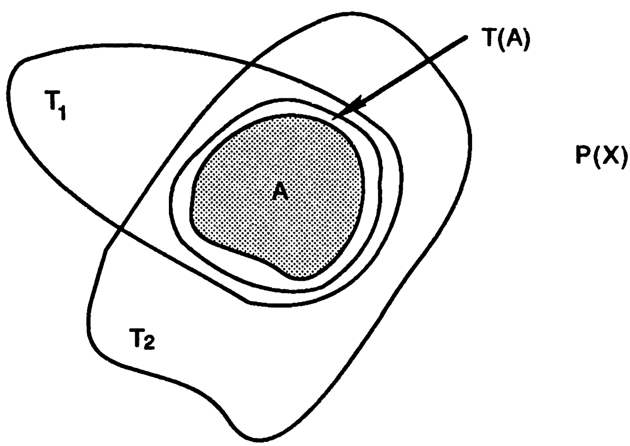

FIGURE 1

Let $T(A)$ be the intersection of all topologies containing A ($P(X)$ is one of such topologies). By Problem 11–8 $T(A)$ is a topology. Since it is the intersection of all topologies containing A, $T(A)$ is the smallest topology, $A \subset T(A)$. By definition $T(A)$ is unique.

We will prove (1). Since $A \subset T(A)$ and $T(A)$ is a topology, $T(A)$ must contain all the sets listed in (1).

On the other hand, the sets listed in (1) form a topology containing A and, therefore, containing $T(A)$.

● PROBLEM 11-19

Let R denote the set of all real numbers and let A denote all the sets of the form

$$\{x : x > a\} \tag{1}$$

$$\{x : x < b\} \tag{2}$$

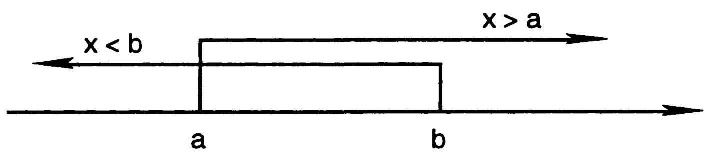

FIGURE 1

Find $T(A)$.

SOLUTION:

By the theorem of Problem 11–18, $T(A)$ consists of

$$T(A) = \{\phi, R, \text{all finite intersections of members of } A,$$
$$\text{all arbitrary unions of finite intersections of members of } A\} \tag{3}$$

All finite intersections of members of A are open intervals (a, b). Hence, we conclude that the

$$\text{set of all open intervals} \subset T(A). \tag{4}$$

The family of all open intervals forms a basis for $T(A)$.

By Problem 11–17 the family of all open intervals forms a basis for the Euclidean topology.

Therefore, $T(A)$ is the Euclidean topology.

● PROBLEM 11-20

R is the set of all real numbers and A consists of all sets of the form

$$\{x : x > a\}$$

$$\{x : x \leq b\} \tag{1}$$

Find $T(A)$ (called the upper limit topology).

SOLUTION:

To find $T(A)$, we must first obtain all finite intersections of elements of A (see Problem 11–18). The sets $(a, b]$

$$(a, b] = \{x : a < x \leq b\} \tag{2}$$

form a basis for topology $T(A)$, which is not Euclidean because $(a, b]$ does not belong to the Euclidean topology.

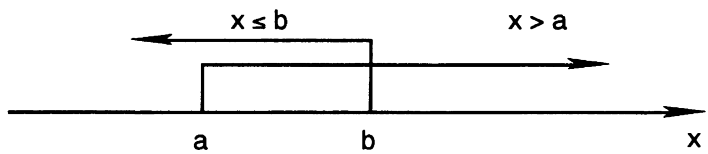

FIGURE 1

Note that we were dealing with the situation, when a family of sets A was given and we had to find topology $T(A)$, such that $A \subset T(A)$. Conversely, for a given topology T, a family of sets $A \subset T$ is called a subbasis for T, if

$$T = T(A).$$

● PROBLEM 11–21

Let $X = \{a, b, c, d, e\}$ and let $A \subset P(X)$ where

$$A = \{\{a, b, c\}, \{a, c, e\}, \{c, d\}\}. \tag{1}$$

Find the topology $T(A)$ on X generated by A.

SOLUTION:

By taking all finite intersections of sets in A and the sets ϕ and X, we find the basis $B(A)$

$$B(A) = \{\phi, X, \{a, b, c\}, \{a, c, e\}, \{c, d\}, \{a, c\}, \{c\}\}. \tag{2}$$

Note that $X \in B$ and $\phi \in B$.

X can be considered the empty intersection of members of A. In order to obtain the topology generated by A, we take all unions of members of $B(A)$.

$$T(A) = \{\phi, X, \{a, b, c\}, \{a, c, e\}, \{c, d\}, \{a, c\}, \{c\}, \{a, b, c, e\},$$
$$\{a, b, c, d\}, \{a, c, e, d\}, \{a, c, d\}\}. \tag{3}$$

● PROBLEM 11-22

Show that a basis determines one and only one topology.

SOLUTION:

Let X represent any set and B a collection of subsets of X. Suppose B is a basis for two topologies T and T', which are different.

Hence, at least one element $A \subset X$ exists, such that $A \in T$ but $A \notin T'$ (or vice versa).

B is a basis for topology T and $A \in T$, therefore

$$A = \bigcup_{\alpha} B_{\alpha} \in T \tag{1}$$

where $B_{\alpha} \in B$ for each α.

Since B is a basis for topology T', any union of elements of B must be an element of T'. Thus,

$$A = \bigcup_{\alpha} B_{\alpha} \in T', \tag{2}$$

which is a contradiction. We conclude that

$$T = T'.$$

A basis as well as a subbasis determines a unique topology.

● PROBLEM 11-23

The following theorem offers a simple method of determining if a given family of sets is a basis.

THEOREM

Let $B = \{B_{\omega} : \omega \in \Omega\}$ denote a family of subsets of X, such that for each $x \in B_{\alpha} \cap B_{\beta}$ and each $\alpha, \beta \in \Omega$ $B_{\gamma} \in B$ exists, such that

$$x \in B_{\gamma} \subset B_{\alpha} \cap B_{\beta} \tag{1}$$

Then $B \cup \{\phi\} \cup \{X\}$ is a basis for some topology $T(B)$ on X. $T(B)$ is unique and is the smallest topology containing B.

■

Prove this theorem.

SOLUTION:

By using B as a subbasis, we obtain a topology $T(B)$. We will show that

B is a basis for topology $T(B)$.

$$T(B) = \{\phi, X, \text{all unions of members of } B\} \tag{2}$$

It is enough to prove that each finite intersection of members of B is a union of members of B. We will prove that for each $x \in B_1 \cap \ldots \cap B_n$, there is a $B_0 \in B$, such that

$$x \in B_0 \subset B_1 \cap \ldots \cap B_n, \tag{3}$$

compare Problem 11–13.

For $n = 2$, (3) is true by the hypothesis. By induction, suppose (3) is true for $n - 1$. Then for n

$$x \in B_1 \cap B_2 \cap \ldots \cap B_{n-1} \cap B_n \tag{4}$$

B_0 exists, such that

$$x \in B_0 \cap B_n \tag{5}$$

$$B_0 \subset B_1 \cap \ldots \cap B_{n-1}. \tag{6}$$

Hence

$$x \in B'_0 \subset B_0 \cap B_n \subset B_1 \cap \ldots B_n \tag{7}$$

for some $B'_0 \in B$.

Thus B is a basis for topology $T(B)$ defined by (2).

● PROBLEM 11–24

> Let $C([0, 1])$ represent the set of all continuous functions on $[0, 1]$. For each $f \in C$ and $r > 0$, we define
>
> $$(f,r) = \{g \in C : \int_0^1 |f - g| < r\}. \tag{1}$$
>
> Show that the family
>
> $$\{K(f, r) : f \in C, r > 0\} \tag{2}$$
>
> forms a basis for some topology T on C.

SOLUTION:

We shall apply the theorem of Problem 11–23. Suppose

$$h \in K(f_1, r_1) \cap K(f_2, r_2). \tag{3}$$

Let

$$t_1 = \int_0^1 \left|f_1 - h\right|, t_2 = \int_0^1 \left|f_2 - h\right| \tag{4}$$

and let

$$R = \min[r_1 - t_1, r_2 - t_2]. \tag{5}$$

Then $R > 0$ and

$$K(h, R)\ \mathrm{A}\ K(f_1, r_1) \cap K(f_2, r_2). \tag{6}$$

Indeed, suppose $F \in K(h, R)$ then

$$\int_0^1 |F - h| < R \tag{7}$$

and

$$\int_0^1 \left|f_1 - F\right| \le \int_0^1 \left|f_1 - h\right| + \int_0^1 \left|h - F\right| < t_1 + (r_1 - t_1) = r_1. \tag{8}$$

Therefore

$$F \in K(f_1, r_1) \quad \text{and} \quad F \in K(f_2, r_2) \tag{9}$$

because

$$\int_0^1 \left|f_2 - F\right| \le \int_0^1 \left|f_2 - h\right| + \int_0^1 \left|h - F\right| < t_2 + (r_2 - t_2) = r_2. \tag{10}$$

● PROBLEM 11-25

Each basis in X yields a unique topology. But distinct bases may yield the same topology.

DEFINITION OF EQUIVALENT BASES

Two bases B and B' in X are equivalent if

$$T(B) = T(B') \tag{1}$$

■

Prove this:

THEOREM

Two bases B and B' in X are equivalent if and only if

1. for each $B_\alpha \in B$ and each $x \in B_\alpha$, there is a $B'_\beta \in B'$, such that

$$x \in B'_\beta \subset B_\alpha$$

and

2. for each $B'_\beta \in B'$ and each $x \in B'_\beta$, there is a $B_\alpha \in B$, such that

$$x \in B_\alpha \subset B'_\beta$$

SOLUTION:

Suppose $$T(B) = T(B'). \tag{2}$$

Then for each $B_\alpha \in B \subset T(B)$ and each $x \in B_\alpha$, since B' is a basis for $T(B)$, by Problem 11–13, there is $B'_\beta \in B'$, such that

$$x \in B'_\beta \subset B_\alpha. \tag{3}$$

Similarly we show that condition 2. holds.

Now suppose condition 1 holds. Since each $A \in T(B)$ is a union of $\{B_\alpha\}$ belonging to B, *then* it follows that $A \in T(B')$. yielding $T(B) \subset T(B')$. Similarly we have $T(B') \subset T(B)$ and therefore $T(B) = T(B')$.

● PROBLEM 11-26

Let S represent a subbase for a topology T on X and let A denote any subset of X. Show that the family

$$S_A = \{A \cap S_\alpha : S_\alpha \in S\} \tag{1}$$

is a subbase for the relative topology T_A on A.

SOLUTION:

Let D denote an open subset of A with respect to topology T_A. Then

$$D = A \cap E \tag{2}$$

where E is a T-open subset of X. Family S is a subbase for T, therefore

$$E = \bigcup (S_{n_1} \cap S_{n_2} \cap \ldots \cap S_{n_k}) \tag{3}$$

where $S_{n_1}, S_{n_2}, \ldots, S_{n_k} \in S$.

Therefore

$$D = A \cap E = A \cap [\bigcup (S_{n_1} \cap \ldots \cap S_{n_k})] =$$

$$= \bigcup [(A \cap S_{n_1}) \cap \ldots \cap (A \cap S_{n_k})]. \tag{4}$$

Thus D is the union of finite intersections of elements of S_A, and S_A is a subbase for T_A.

1. Let $A =]0, 1]$ and $B = \{{}^1/_n : n \in N\}$ be subsets of R. Find the closure of A and B.

2. Find the closure of 0 and 1 in Sierpinski space, where $X = \{0, 1\}$ and $T = \{\phi, X, 0\}$.

SOLUTION:

DEFINITION OF A NEIGHBORHOOD

Let (X, T) denote a topological space. A neighborhood of an $x \in X$ is any open set N_0 containing x, $x \in N_0$. A neighborhood of x is denoted by $N_0(x)$.

■

DEFINITION OF ADHERENT POINT

Let $A \subset X$. A point $x \in X$ is adherent to A, if for each $N_0(x)$

$$N_0(x) \cap A \neq \phi. \tag{1}$$

■

DEFINITION OF THE CLOSURE OF THE SET

The closure of A, denoted by $\overline{A}$, is the set of all points in X adherent to A.

$$\overline{A} = \{x \in X : \forall\, N_0(x) : N_0(x) \cap A \neq \phi\}. \tag{2}$$

■

1. The closure of $A =]0, 1]$ is $\overline{A} = [0, 1]$. The closure of $B = \{{}^1/_n : n \in N\}$is

$$\overline{B} = \{0, {}^1/_n\} = \{0\} \cup B \tag{3}$$

2. The open sets are ϕ, X and 0. We have

$$\overline{0} = \{0, 1\} = X$$

$$\overline{1} = 1 \tag{4}$$

● PROBLEM 11-28

Show that:

1. for every set A, $A \subset \overline{A}$.

2. A is closed iff $A = \overline{A}$.

SOLUTION:

DEFINITION

A set $A \subset X$ is called closed if $X - A$ is an open set.

■

1. Obviously, if $x \in A$, then

$$N_0(x) \cap A \neq \phi. \tag{1}$$

Hence
$$x \in \overline{A} \qquad A \subset \overline{A}. \tag{2}$$

2. (A closed) $\Rightarrow$ ($A = \overline{A}$).

If A is closed, then $X - A$ is open. Each $x \notin A$ has a neighborhood $N_0(x)$, such that

$$N_0(x) \cap A \neq \phi. \tag{3}$$

Hence
$$x \notin \overline{A} \text{ and } \overline{A} \subset A. \tag{4}$$

From (4) and (2), we obtain $A = \overline{A}$.

($A = \overline{A}$) $\Rightarrow$ (A closed).

Since $A = \overline{A}$, each $x \notin$ A has a $N_0(x)$, such that

$$N_0(x) \cap A \neq \phi. \tag{5}$$

Hence, $X - A$ is open and A is closed.

● PROBLEM 11-29

Show that this is an alternative definition of a closed set:

$\overline{A}$ is the smallest closed set containing A; i.e.,

$$\overline{A} = \cap \{D : (D \text{ is closed}) \wedge (A \subset D)\}. \tag{1}$$

SOLUTION:

$$\overline{A} \subset \cap D. \tag{2}$$

Suppose $x \in \overline{A}$ and $x \notin \bigcap D$. Since each set D is closed, $\bigcap D$ is a closed set

$$(x \notin \bigcap D) \Rightarrow x \in X - \bigcap D, \text{ where } X - \bigcap D \text{ is an open set} \Rightarrow$$
$$\Rightarrow x \text{ has a nbd } G, \text{ such that } G \cap \bigcap D = \phi \Rightarrow$$
$$\Rightarrow G \cap A = \phi \text{ (because } A \subset \bigcap D) \Rightarrow x \notin \overline{A}. \tag{3}$$

Now we will prove that $\bigcap D \subset \overline{A}$. Suppose $x \notin \overline{A}$, then there is a neighborhood of x, $N_0(x)$, such that

$$N_0(x) \cap A \neq \phi. \tag{4}$$

Therefore, $X - N_0(x)$ is closed and contains A. Thus $X - N_0(x)$ is one of the sets (denoted by D) in (1).

Since

$$x \notin X - N_0(x) \quad \text{also} \quad x \notin \bigcap D. \tag{5}$$

See Figure 1.

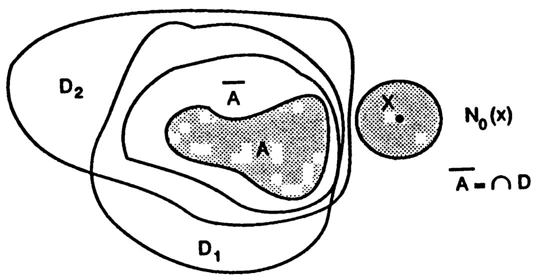

FIGURE 1

● PROBLEM 11-30

Prove that:

1. $\overline{\overline{A}} = \overline{A}$.

2. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$.

3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

4. $\overline{\phi} = \phi$.

SOLUTION:

1. $\overline{A} = \bigcap D$ (where D is closed). Since the intersection of any family of closed sets is a closed set, $\overline{A}$ is a closed set. A is closed if and only if $A = \overline{A}$. Hence

$$\overline{\overline{A}} = \overline{A}. \tag{1}$$

2. $\overline{A} = \bigcap D$ and $\overline{B} = \bigcap D'$, where each D contains A and each D' contains B. Since $A \subset B$ each D' contains B and contains A. Hence,

$$\overline{A} \subset \overline{B}. \tag{2}$$

3. Set $\overline{A \cup B}$ is closed. Since for each A, $A \subset \overline{A}$, we obtain

$$\overline{A} \cup \overline{B} \subset \overline{A \cup B}. \tag{3}$$

On the other hand, set $\overline{A} \cup \overline{B}$ is closed and

$$A \cup B \subset \overline{A} \cup \overline{B}. \tag{4}$$

Hence

$$\overline{A \cup B} \subset \overline{A} \cup \overline{B}. \tag{5}$$

From (3) and (5) we get

$$\overline{A \cup B} = \overline{A} \cup \overline{B}. \tag{6}$$

4. Set ϕ is a closed set, hence

$$\overline{\phi} = \phi. \tag{7}$$

• PROBLEM 11-31

1. Find the derived set of

$A = \{^1/_n : n \in N\}$

$B =]0, 1]$

$C = \{x : x \in (0,1), x \text{ is a rational number}\}$.

2. Let X denote an indiscrete topological space. Find the derived set A' of any subset $A \subset X$.

SOLUTION:

DEFINITION OF CLUSTER POINT

Let $A \subset X$. A point $x \in X$ is called a cluster point of A if each neighborhood of x, N_0 of x contains at least one point of A distinct from x.

■

It is easy to see that 0 is the only cluster point of $\{^1/_n\}$.

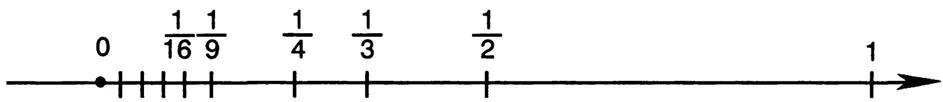

FIGURE 1

Indeed each neighborhood of 0 contains at least one (as a matter of fact, infinitely many) point of $A = \{^1/_n\}$ distinct from 0.

DEFINITION OF A DERIVED SET

The set

$$A' = \{x \in X : \forall\, 0(x) : 0(x) \cap (A - x) \neq \phi\} \tag{1}$$

of all cluster points of A is called the derived set of A.

■

1. We obtain

$A' = \{0\}$

$B' = [0, 1]$

$C' = [0, 1]$.

2. We have $T = (X, \phi)$. For any point $x \in X$, X is the only open set containing x. We have

$$A' = \begin{cases} \phi & \text{if } A = \phi \\ X & \text{if } A \text{ contains two or more points} \\ X - \{a\} & \text{if } A = \{a\} \end{cases}$$

● PROBLEM 11-32

Show that

$$\overline{A} = A \cup A'. \tag{1}$$

Set A is closed if and only if $A' \subset A$.

SOLUTION:

Suppose $x \in A'$ then $x \in \overline{A}$. Hence $A' \subset \overline{A}$. Also $A \subset \overline{A}$, therefore

$$A' \cup A \subset \overline{A}. \tag{2}$$

To prove the converse inclusion suppose $x \in \overline{A}$. If $x \in A$, then the proof is finished.

If $x \notin A$, then each neighborhood of x intersects A at a point distinct from x, hence $x \in A'$. Thus

$$\overline{A} \subset A \cup A'. \tag{3}$$

We conclude that

$$\overline{A} = A \cup A'. \tag{4}$$

Set A is closed if and only if $A = \overline{A}$.

Thus, A is closed if and only if

$$A' \subset A \tag{5}$$

i.e., if and only if A contains all of its cluster points.

● PROBLEM 11-33

Prove that

$$(A \cup B)' = A' \cup B'. \tag{1}$$

SOLUTION:

Observe that if $A \subset B$, then $A' \subset B'$. Since $A \subset A \cup B$ and $B \subset A \cup B$, we obtain

$$A' \subset (A \cup B)' \qquad B' \subset (A \cup B)' \tag{2}$$

Thus

$$A' \cup B' \subset (A \cup B)'. \tag{3}$$

Now we shall prove the inverse inclusion, $(A \cup B)' \subset A' \cup B'$. Suppose, $a \notin A' \cup B'$, then open sets 0_1 and 0_2 exist, such that

$$a \in 0_1 \quad \text{and} \quad a \in 0_2 \tag{4}$$

$$0_1 \cap A \subset \{a\}, \quad 0_2 \cap B \subset \{a\}. \tag{5}$$

Since sets 0_1 and 0_2 are open, $0_1 \cap 0_2$ is open and $a \in 0_1 \cap 0_2$. We obtain

$$(0_1 \cap 0_2) \cap (A \cup B) = (0_1 \cap 0_2 \cap A) \cup (0_1 \cap 0_2 \cap B)$$

$$\subset (0_1 \cap A) \cup (0_2 \cap B) \subset \{a\} \cup \{a\} = \{a\}. \tag{6}$$

Hence

$$a \notin (A \cup B)' \quad \text{and} \quad (A \cup B)' \subset A' \cup B'. \tag{7}$$

● **PROBLEM 11-34**

1. Find the interior of the sets

   $A = \{{}^1/_n : n \in N\}$

   $B = [0, 1]$

2. Prove that

$$Int(A) = X - \overline{(X - A)} \tag{1}$$

SOLUTION:

DEFINITION OF THE INTERIOR

The interior $Int(A)$ of $A \subset X$ is the largest open set contained in A, that is

$$Int(A) = \bigcup \{0 : (0 \text{ open}) \wedge (0 \subset A)\}. \tag{2}$$

■

1. $Int(A) = \phi$.

   $Int(B) = (0, 1)$

2. We shall apply

$$A \subset B \Leftrightarrow X - B \subset X - A. \tag{3}$$

Therefore, if $0 \subset A$, then

$$X - A \subset X - 0 = D. \tag{4}$$

Since 0 is an open set, D is a closed set. We obtain

$$Int(A) = \bigcup \{X - D : (D \text{ closed}) \wedge (X - A \subset D)\} =$$

$$= X - \bigcap \{D : (D \text{ closed}) \wedge (X - A \subset D)\} = X - \overline{(X - A)}. \tag{5}$$

● **PROBLEM 11-35**

Find the boundary of the sets

$A = \{{}^1/_n : n \in N\}$

$B = (0, 1]$

Show that

$$\mathcal{F}r\,(A) = \overline{A} - Int(A). \tag{1}$$

SOLUTION:

DEFINITION OF BOUNDARY

The boundary of a set $A \subset X$ is denoted by $\mathcal{F}r(A)$ and defined by

$$\mathcal{F}r(A) = \overline{A} \cap \overline{(X - A)}. \tag{2}$$

■

We have

$$\overline{A} = A \cup \{0\} \quad \text{and} \quad \overline{(X - A)} = R.$$

Thus

$$\mathcal{F}r(A) = A \cup \{0\} \quad \overline{B} = [0,1]$$

$$\overline{(X - B)} = (-\infty, 0] \cup [1, \infty) \tag{3}$$

Hence

$$\mathcal{F}r(B) = \overline{B} \cap \overline{(X - B)} = \{0, 1\}. \tag{4}$$

We have

$$\mathcal{F}r(A) = \overline{A} \cap \overline{(X - A)}. \tag{5}$$

But

$$Int(A) = X - \overline{(X - A)}. \tag{6}$$

Also note that

$$X - (X - B) = B \tag{7}$$

Hence

$$\mathcal{F}r(A) = \overline{A} \cap X - [X - \overline{(X - A)}] =$$

$$= \overline{A} \cap [X - Int(A)] = \overline{A} - Int(A) \tag{8}$$

● PROBLEM 11-36

> Show that the following statements are equivalent:
>
> 1. D is dense in X.
> 2. If G is a closed set and $D \subset G$, then $G = X$.
> 3. Each nonempty basic open set in X contains an element of D.
> 4. The complement of D has empty interior.

SOLUTION:

DEFINITION OF A DENSE SET

Set $D \subset X$ is dense in X if $\overline{D} = X$.

■

1. $\Rightarrow$ 2.

Set D is dense in X, $\overline{D} = X$. $D \subset G$, then $\overline{D} \subset \overline{G}$, but $\overline{G} = G$, hence $X \subset G$ and

$$X = G. \tag{1}$$

2. $\Rightarrow$ 3.

Let $0 \neq \phi$ be open and $0 \cap D = \phi$, then

$$D \subset X - 0 \neq X \tag{2}$$

in contradiction with 2 because $X - 0$ is closed.

3. $\Rightarrow$ 4.

Suppose $Int(X - D) = \phi$. Set $Int(X - D)$ is open, hence, there is a nonempty basic set

$$A \subset Int(X - D). \tag{3}$$

But

$$Int(X - D) \subset X - D. \tag{4}$$

Then $A \subset X - D$ and A contains no points of D.

4. $\Rightarrow$ 1.

$$Int(X - D) = X - \overline{[X - (X - D)]} = X - \overline{D} = \phi. \tag{5}$$

Thus $$\overline{D} = X.$$

● PROBLEM 11–37

Set X is given. To promote it to the topological space (X, T), we have to define a family T of its subsets which satisfies certain conditions. There are many ways of defining (or finding) a topology on X. For example, a function f, which assigns its closure to every subest of X determines a topology

$$f : A \rightarrow \overline{A}. \tag{1}$$

Without getting into details, we want the closure $f(A)$ of the set to satisfy these conditions:

$$A \subset \overline{A} \qquad \text{(See Problem 11–28)}$$

$$\overline{\phi} = \phi, \ \overline{\overline{A}} = \overline{A}, \ \overline{A \cup B} = \overline{A} \cup \overline{B} \qquad \text{(See Problem 11–30)}$$

Prove the following:

THEOREM

Let X represent a set and $f : P(X) \rightarrow P(X)$ a function, such that

1. $f(\phi) = \phi$.
2. For each $A, A \subset f(A)$.

3. $f \circ f(A) = f(A)$ for each A.

4. $f(A \cup B) = f(A) \cup f(B)$ for each A, B.

The family

$$T = \{X - f(A) : A \in P(X)\} \tag{2}$$

is a topology and $\overline{A} = f(A)$ for each A.

■

Prove this theorem.

SOLUTION:

Note that

$$(A \subset B) \Rightarrow (f(A) \subset f(B)). \tag{3}$$

Indeed if $A \subset B$, then $A \cup B = B$ and $f(A \cup B) = f(A) \cup f(B) = f(B)$. Now we will show that T is a topology .

I. $\phi, X \in T$.
Since $X \subset f(X)$, we get $X = f(X)$. Hence $\phi \in T$. Since $X - f(\phi) = X, X \in T$.

II. Suppose $X - f(A)$ and $X - f(B)$ belong to T. Then

$$[X - \mathrm{f}(A)] \cap [X - f(B)] = X - [f(A) \cup f(B)] = X - f(A \cup B) \in T. \tag{4}$$

Thus, intersection of any finite family of sets of T is a member of T.

III. Let $S = \bigcup_\alpha X - f(A_\alpha)$.We will show that for some $U \in P(X)$, $S = X - f(U)$. We have

$$S = X - \bigcap_\alpha f(A_\alpha) \tag{5}$$

and

$$X - S = \bigcap_\alpha f(A_\alpha) \subset f(A_\alpha) \tag{6}$$

for each α.

By (3) and condition 3, we obtain

$$f(X - S) \subset f \circ f(A_\alpha) = f(A_\alpha) \tag{7}$$

for each α. Thus

$$f(X - S) \subset \bigcap_\alpha f(A_\alpha) = X - S. \tag{8}$$

from (8) and condition 2, we obtain

Prove the following convenient way of describing F_σ and G_δ sets:

$$X - S = f(X - S). \tag{9}$$

Hence

$$S = X - f(X - S) = X - f(U). \tag{10}$$

The union of any family of members of T is again a member of T.

By using T as the topology on X, we will show that $\overline{A} = f(A)$. Indeed, $\overline{A} \subset f(A)$ because each $f(A)$ is closed in T and $A \subset f(A)$, we obtain $\overline{A} \subset \overline{f(A)} = f(A)$. Similarly $f(A) \subset \overline{A}$. Since $X - \overline{A}$ is open in T, for some B, $f(B) = \overline{A}$. Since

$$A \subset \overline{A} \tag{11}$$

we obtain

$$f(A) \subset f(\overline{A}) = f \circ f(B) = f(B) = \overline{A}. \tag{12}$$

That completes the proof.

• PROBLEM 11-38

Show that in R the closed interval $[a, b]$ is an F_γ set and also a G_δ set. Show that the set of rational numbers in R is F_γ.

SOLUTION:

DEFINITION OF F_σ AND G_δ SETS

Set F is called an F_σ set if it is the union of at most countably many closed sets. A set G is called a G_δ set if it is the intersection of at most countably many open sets.

■

A closed interval is a closed set, hence, it is an F_σ. On the other hand,

$$[a,b] = \bigcap_{n=1} (a - {}^1/_n, b + {}^1/_n) \tag{1}$$

$[a, b]$ is the intersection of countably many open sets. Hence, $[a, b]$ in R is a G_δ.

The set of rational numbers Q is countable and since each point in R is a closed set, Q is the union of countably many closed sets. Thus, Q is an F_γ.

• PROBLEM 11-39

Prove the following convenient way of describing F_σ and G_δ sets:

1. If F is an F_γ, then there is a non-decreasing sequence of closed sets

$$F_1 \subset F_2 \subset F_3 \subset \ldots \text{ where } F = \bigcup_1^\infty F_n. \tag{1}$$

2. If G is a G_δ, then there is a non-increasing sequence of open sets

$$G_1 \supset G_2 \supset G_3 \ldots \text{ where } G = \bigcap_1^\infty G_n. \tag{2}$$

SOLUTION:

1. F is an F_γ. Hence

$$F = \bigcup_1^\infty A_n \text{ , } A_n \text{ closed.} \tag{3}$$

F is the union of at most countably many closed sets. Let

$$F_1 = A_1, F_2 = A_1 \cup A_2, F_3 = A_1 \cup A_2 \cup A_3, \ldots,$$

$$\ldots, F_n = A_1 \cup \ldots \cup A_n. \tag{4}$$

Then sets F_n are closed and

$$\bigcup_1^\infty A_n = \bigcup_1^\infty F_n, \; F_1 \subset F_2 \subset F_3 \ldots. \tag{5}$$

2. Similarly, if G is a G_δ, then

$$G = \bigcap_1^\infty B_n \tag{6}$$

where B_n are open sets. Let

$$G_1 = B_1, G_2 = B_1 \cap B_2, \ldots, G_n = B_1 \cap \ldots \cap B_n. \tag{7}$$

Then each G_n is open and

$$G_1 \supset G_2 \supset G_3 \supset \ldots \bigcap_1^\infty B_n = \bigcap_1^\infty G_n. \tag{8}$$

● PROBLEM 11–40

Prove that:

1. The countable union and finite intersection of F_γ sets is an F_σ.

2. The countable intersection and finite union of G_δ sets is a G_δ.

SOLUTION:

1. Consider the countable union of F_ρ sets F_σ

$$F_i = \bigcup_{k=1}^{\infty} F_{i,k} \tag{1}$$

where $F_{i,k}$ are closed sets. The countable union of F_i's is

$$\bigcup_{i=1}^{\infty}\left[\bigcup_{k=1}^{\infty} F_{i,k}\right] = \bigcup_{i,j}^{\infty}\left\{F_{i,j} : (i,j) \in N \times N\right\}. \tag{2}$$

Set $N \times N$ is countable and $F_{i,j}$ are closed sets, hence, the union is an F_γ. The finite intersection is

$$\bigcap_{i=1}^{n}\bigcup_{k=1}^{\infty} F_{i,k} = \left[\bigcup_{k=1}^{\infty} F_{1,k}\right] \cap \left[\bigcup_{k=1}^{\infty} F_{2,k}\right] \cap \ldots \cap \left[\bigcup_{k=1}^{\infty} F_{n,k}\right] =$$

$$= \bigcup\left\{F_{1,k_1} \cap \ldots \cap F_{n,k_n} : (k_1, \ldots, k_n) \in N \times \ldots \times N\right\}. \tag{3}$$

Each set $F_{1,k_1} \cap \ldots \cap F_{n,k_n}$ is closed and (3) is a countable union.

2. Let

$$G_i = \bigcap_{k=1}^{\infty} G_{i,k}\ ;\ i = 1, 2, 3, \ldots \tag{4}$$

where $G_{i,k}$ are open sets, be a family of G_δ sets. The countable intersection is

$$\bigcap_{i=1}^{\infty}\bigcap_{k=1}^{\infty} G_{i,k} = \bigcap_{i,k}\left\{G_{i,k} : (i,k) \in N \times N\right\} \tag{5}$$

Sets $G_{i,k}$ are open and $N \times N$ is a countable set. Hence, (5) is a G_δ set. Consider now the finite union of G_i's

$$\bigcup_{i=1}^{n}\bigcap_{k=1}^{\infty} G_{i,k} = \bigcap\{G_{1,k_1} \cup G_{2,k_2} \cup \ldots \cup G_{n,k_n} :$$
$$(k_1, k_2, \ldots, k_n) \in N \times N \times \ldots \times N\}. \tag{6}$$

(6) is a G_δ set.

● PROBLEM 11-41

1. Show that the complement of an F_γ is a G_σ.

2. Show that the complement of a G_δ is an F_γ.

SOLUTION:

1. Let F be an F_γ set, then

$$F = \bigcup_{n=1}^{\infty} F_n \tag{1}$$

where F_n are closed sets. The complement of F is

$$X - F = X - \bigcup_{n=1}^{\infty} F_n. \tag{2}$$

By applying DeMorgan's rule, we find

$$X - F = \bigcap_{n=1}^{\infty} (X - F_n). \tag{3}$$

Since each F_n is closed, $X - F_n$ is open for each n. Thus $X - F$ is the intersection of at most countably many open sets.

2. Let G be a G_δ set. Then

$$G = \bigcap_{n=1}^{\infty} G_n \tag{4}$$

where G_n are open sets. The complement of G is

$$X - G = X - \bigcap_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} (X - G_n) \tag{5}$$

where $X - G_n$ are closed sets. Hence $X - G$ is an F_σ set.

● PROBLEM 11-42

Show that each F_σ and each G_δ is a Borel set.

SOLUTION:

DEFINITION OF A σ-RING

A nonempty family of sets

$$Q \subset P(X) \tag{1}$$

is called a σ-ring if

1. $A \in Q \Rightarrow X - A \in Q$
2. $\forall\, n : A_n \in Q \Rightarrow \bigcup_{1}^{\infty} A_n \in Q.$

■

DEFINITION OF BOREL SETS

If (X, T) is a topological space, then a unique smallest σ-ring B containing the topology T of X exists, that is $T \subset B$. Family of sets B is called the family of Borel sets in X.

■

Let F denote an F_σ–set, then

$$F = \bigcup_1^\infty F_n \tag{2}$$

where F_n are closed sets. The sets $X - F_n$ are open sets. Since B is a σ-ring such that $T \quad B$, each open set belongs to B. Hence

$$\forall n : X - F_n \in B \tag{3}$$

$$(X - F_n \in B) \Rightarrow (X - (X - \mathrm{F_n}) \in B) \tag{4}$$

Thus

$$\forall n : F_n \in B \text{ and } F = \bigcup_1^\infty F_n \in B \tag{5}$$

If G is a G_δ–set then

$$G = \bigcap_1^\infty G_n \tag{6}$$

where G_n are open sets. But

$$\bigcap_1^\infty G_n = X - \left[\bigcup_1^\infty (X - G_n)\right] \tag{7}$$

Each G_n is an open set, therefore

$$G_n \in B \Rightarrow X - G_n \in B \Rightarrow \bigcup_1^\infty (X - G_n) \in B \Rightarrow$$

$$\Rightarrow X - \left[\bigcup_1^\infty (X - G_n)\right] \in B \Rightarrow G \in B. \tag{8}$$

• PROBLEM 11-43

Show that the countable union, countable intersection, and the difference of Borel sets is a Borel set.

SOLUTION:

Let B denote the family of Borel sets in X. B can be defined as the intersection of all σ–rings containing T.

It is easy to verify that the intersection of any family of σ-rings is a σ-ring. Thus, B is a σ-ring.

$$\left(B_i \in B \text{ for } i = 1, 2, 3, \ldots\right) \Rightarrow \left(\bigcup_1^\infty B_i \in B\right) \tag{1}$$

Because B is a σ-ring.

$$\left(B_i \in B \text{ for } i = 1, 2, \ldots\right) \Rightarrow \left(X - B_i \in B \text{ for } i = 1, 2, \ldots\right) \Rightarrow$$

$$\Rightarrow \left(\bigcup_1^\infty (X - B_i) \in B\right) \Rightarrow \left(X - \bigcup_1^\infty (X - B_i) = \bigcap_1^\infty B_i \in B\right). \tag{2}$$

$$(A_1, A_2 \in B) \Rightarrow (A_1, X - A_2 \in B) \Rightarrow$$

$$\Rightarrow (A_1 \cap (X - A_2) = A_1 - A_2 \in B). \tag{3}$$

● PROBLEM 11-44

Show that in the Euclidean space R^n there are sets that are not Borel sets.

SOLUTION:

We shall apply the following:

THEOREM

Let (X, T) be a topological space and B be the family of Borel sets in X. Then

$$\text{card } (B) \leq \text{card } (T)^{\text{card N}} \tag{1}$$

■

Topological space R^n has a countable basis. Hence, the cardinal number of the Euclidean topology of R^n is $2^{\text{card } N}$

$$\text{card } (T) = 2^{\text{card } N}. \tag{2}$$

From (1) and (2) we find

$$\text{card } (B) \leq 2^{\text{card } N}. \tag{3}$$

Since

$$\text{card } (P(R^n)) = 2^{\text{card } R} \tag{4}$$

we conclude that there are sets in R^n that are not Borel sets.

● PROBLEM 11-45

Let (R, T) denote Euclidean space and X the subset of R

$$X = (0, 1] \cup \{2\}. \tag{1}$$

Find the induced topology T_X.

SOLUTION:

DEFINITION OF SUBSPACE

Let (X, T) denote a topological space and $Y \subset X$. The induced topology T_Y on Y is defined by

$$T_Y =: \{Y \cap T_\alpha : T_\alpha \in T\}. \tag{2}$$

The space (Y, T_Y) is called a subspace of (X, T).

■

It is easy to show that T_Y defined in (2) is a topology on Y. Consider (R, T). T is the family of all open sets in R. By definition

$$T_X = \{[\,(0, 1] \cup \{2\}] \cap T_\alpha : T_\alpha \in T\} \tag{3}$$

where T_α are open sets.

The family T_X consists of

1. $\{2\}$
2. all open intervals contained in $(0, 1]$
3. all intervals of the form $]a, 1]$ where $a \in (0, 1)$.

Interval $(0, 1]$ belongs to T_X, hence it is an open set. Since $\{2\} \in T_X$, interval $(0, 1]$ is a closed set in X.

● PROBLEM 11-46

Let (X, T) denote a space and (Y, T_Y) a subspace. Show that if $\{B_\omega : \omega \in \Omega\}$ is a basis for T, then $\{Y \cap B_\omega : \omega \in \Omega\}$ is a basis for T_Y.

SOLUTION:

Let (X, T) denote a topological space and $\{B_\omega : \omega \in \Omega\}$ be its basis. Then each open set in X is the union of members of $\{B_\omega\}$. The induced topology T_Y is defined by

$$T_Y = \{Y \cap T_\alpha\}. \tag{1}$$

Suppose A is an open set in Y, $A \in T_Y$. Then

$$A = Y \cap T_\alpha \tag{2}$$

for some α. But $\{B_\omega\}$ is a basis for T,

$$T_\alpha = U\, B_\omega. \tag{3}$$

Thus

$$A = Y \cap [U\, B_\omega] = U(Y \cap B_\omega) \tag{4}$$

Therefore $\{Y \cap B_\omega : \omega \in \Omega\}$ is a basis for T_Y.

● PROBLEM 11-47

Let (X, T) denote a topological space and (Y, T_Y) be a subspace. Show that a set $A \subset Y$ is T_Y-closed if and only if $A = Y \cap D$, where D is T-closed.

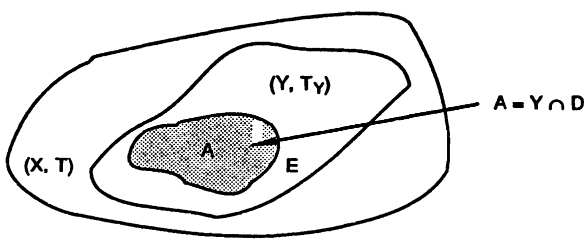

FIGURE 1

SOLUTION:

We will show that the closed sets in Y are the intersections of Y and the closed sets in X. Suppose $A \subset Y$ is closed in Y, then $A = Y - E$, where E is open in Y. Therefore

$$E = Y \cap G \tag{1}$$

where $G \in T$. Hence

$$A = Y - E = Y - Y \cap G = Y \cap (X - G). \tag{2}$$

Since $G \in T$, $X - G$ is T-closed.

Now, suppose

$$A = Y \cap D \tag{3}$$

where D is T-closed. Then

$$Y - A = Y - Y \cap D = Y \cap (X - D) \tag{4}$$

where $X - D$ is T-open and $Y - A$ is T_Y-open. Thus A is T_Y-closed.

● PROBLEM 11–48

1. (X, T) is a topological space and (Y, T_Y) is its subspace. In Problem 11–45, we showed that sets, open in a subspace need not be open in the entire space. Prove the following:

THEOREM

Let (X, T) denote a space and (Y, T_Y) its subspace. If $A \subset Y$ is open (closed) in Y, and Y is open (closed) in X, then A is open (closed) in X.

2. Is a subspace of a subspace a subspace of the entire space?

SOLUTION:

1. Suppose A is open in Y. Then

$$A = Y \cap G \tag{1}$$

where G is open in X. Since Y is open in X, $Y \cap G$ is open in X. The same reasoning holds for closed sets.

2. Yes. Because if

$$Z \subset Y \subset X \tag{2}$$

and T_{YZ} is the topology of Z with respect to Y, then

$$T_{YZ} = T_Z \tag{3}$$

Where T_Z is the topology of Z with respect to X. We shall prove (3). Let

$$U \in T_{YZ} \tag{4}$$

then

$$U = Z \cap V \tag{5}$$

where $V \in T_Y$. But

$$V = Y \cap Q \tag{6}$$

where $Q \in T$. Then

$$U = Z \cap V = Z \cap Y \cap Q = Z \cap Q. \tag{7}$$

Hence $U \in T_Z$ and $T_{YZ} \subset T_Z$.

The converse inclusion is obvious.

CHAPTER 12

CONTINUITY, HOMEOMORPHISMS, AND TOPOLOGICAL EQUIVALENCE

HOMEOMORPHIC SPACES

IDENTIFICATION SPACES

Let (X, T_X) and (Y, T_Y) represent topological spaces, such that

$X = \{a, b, c, d\}$

$T_X = \{\phi, X, \{a\}, \{a, b\}, \{a, b, c\}\}$

$Y = \{x, y, z, w\}$

$T_Y = \{\phi, Y, \{y\}, \{y, z, w\}$

Which of the functions depicted in Figure 1 is continuous?

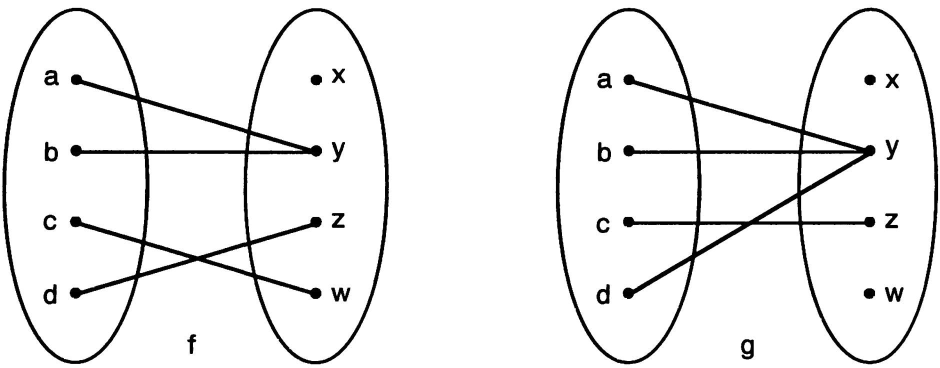

FIGURE 1

SOLUTION:

DEFINITION OF CONTINUOUS FUNCTIONS

Let (X, T_X) and (Y, T_Y) represent topological spaces. A function $f : X \to Y$ is called continuous if the inverse image of each open set in Y is an open set in X. That is, if $f : X \to Y$ is continuous, then

$$f^{-1} : T_Y \to T_X. \tag{1}$$

■

Function f is continuous because f^{-1} maps each open set in T_Y into an open set in T_X.

$$f^{-1}(\phi) = \phi \in T_X$$

$$f^{-1}(Y) = X \in T_X$$

$$f^{-1}(\{y\}) = \{a, b\} \in T_X$$

$$f^{-1}(\{y, z, w\}) = X \in T_X. \tag{2}$$

Function g is not continuous, because

$$\{y\} \in T_Y \text{ but } g^{-1}(\{y\}) = \{a, b, d\} \notin T_X. \tag{3}$$

● PROBLEM 12-2

Show that the identity function

$$f: (X, T) \to (X, T_1) \tag{1}$$

where $\forall\, x \in X: f(x) = x$, is continuous if and only if T is finer than T_1, that is, if

$$T_1 \subset T. \tag{2}$$

SOLUTION:

Suppose f is continuous then

$$A \in T_1 \Rightarrow f^{-1}(A) \in T \tag{3}$$

But $f^{-1}(A) = A$. Hence $T_1 \subset T$.

Now suppose $f: (X, T) \to (X, T_1)$ is the identity function, $f(x) = x$ and suppose $T_1 \subset T$.

Let $A \in T_1$. Then $f^{-1}(A) = A \in T$ because $T_1 \subset T$. Thus, f is continuous.

■

Consider the sequence of identity functions

$$(X, T_1) \xrightarrow{\text{identity}} (X, T_2) \xrightarrow{\text{identity}} (X, T_3) \xrightarrow{\text{identity}} (X, T_4) \longrightarrow (\ldots$$

To ensure continuity of the functions, we must have

T_1 is finer than T_2
T_2 is finer than T_3
T_3 is finer than T_4.

● PROBLEM 12-3

1. Show that

$$f: (X, P(X)) \to (Y, T_Y) \tag{1}$$

is always continuous.

2. Show that

$$f:(X, T_X) \to (Y, \{\phi, Y\}) \tag{2}$$

is always continuous.

SOLUTION:

1. Let us take any open subset in T_Y, $A \in T_Y$. Then $f^{-1}(A)$ is an open set in X because $T_X = P(X)$, i.e., each subset of X is an open set.

2. There are only two open sets in Y, ϕ and Y.

$$f^{-1}(\phi) = \phi \tag{3}$$

which is an open set in X, $\phi \in T_x$. Similarly

$$f^{-1}(Y) = X \in T_X. \tag{4}$$

Each topology in X contains ϕ and X. Hence f defined by (2) is always continuous.

● PROBLEM 12-4

Suppose

$$f:(X, T_X) \to (Y, T_Y) \tag{1}$$

is not a continuous function. Show that the same function

$$f:(X, T'_X) \to (Y, T'_Y) \tag{2}$$

is also not continuous if

$$T'_X \text{ is coarser than } T_X \text{ (that is } T'_X \subset T_X)$$

and

$$T'_Y \text{ is finer than } T_Y \text{ (that is } T_Y \subset T'_Y).$$

SOLUTION:

Since

$$f:(X, T_X) \to (Y, T_Y)$$

is not continuous, there is an open set $D \in T_Y$, such that

$$f^{-1}(D) \notin T_X. \tag{3}$$

Consider function (2). Since $D \in T_Y$ and $T_Y \subset T'_Y$, we have

$$D \in T'_Y. \tag{4}$$

Hence $f^{-1}(D) \notin T_X$ and since $T'_X \subset T_X$

$$f^{-1}(D) \notin T'_X. \tag{5}$$

Function defined in (2) is not continuous.

Note that the same will be true if we replace T'_X by T_X in equation (2).

● PROBLEM 12-5

Show that the function

$$f : R \to R$$

defined by

$$f(x) = \begin{cases} x & \text{if } \; x \le 1 \\ x + 1 & \text{if } \; x > 1 \end{cases} \tag{1}$$

is not continuous if R is equipped with the Euclidean topology but becomes continuous when R has the upper limit topology.

SOLUTION:

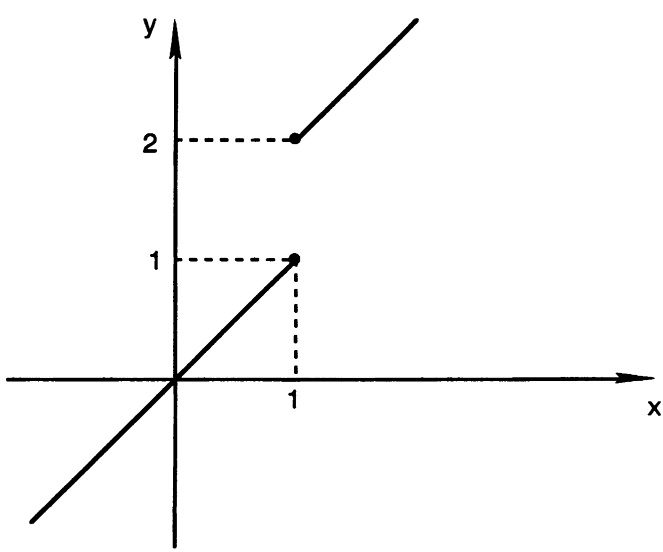

FIGURE 1

The Euclidean topology consists of all open intervals and the unions of open intervals.

Let $A = (0, {}^3/_2)$. Then

$$f^{-1}(A) = f^{-1}(0, {}^3/_2) =]0, 1]. \tag{2}$$

The inverse image of an open set is not an open set. Hence f is not continuous in the Euclidean topology.

The upper limit topology on R consists of unions of open-closed intervals, that is intervals of the form $]a, b]$.

It is easy to verify that for each $]a, b]$

$$f^{-1}[(a, b]] =]c, d]. \tag{3}$$

The inverse image of an open set is an open set, and the function f is continuous.

● PROBLEM 12-6

Function $f : X \to Y$ is continuous if and only if the inverse image of every closed subset of Y is a closed subset of X.

Prove it.

SOLUTION:

$$(f : X \to Y \text{ is continuous}) \Leftrightarrow$$
$$\Leftrightarrow (\forall\, D \subset Y : D \text{ closed in } Y f^{-1}(D) \text{ closed in } X)$$

$\Rightarrow$ Suppose $f : X \to Y$ is continuous and $D \subset Y$ is any closed subset of Y. Then $Y - D$ is an open subset of Y and

$$f^{-1}(Y - D) \tag{1}$$

is an open subset of X.

But

$$f^{-1}(Y - D) = X - f^{-1}(D). \tag{2}$$

Hence $f^{-1}(D)$ is closed.

$\Leftarrow$ Suppose for every $D \subset Y$, D is closed in Y, $f^{-1}(D)$ is closed in X. Let A represent an open subset of Y. Then $Y - A$ is closed in Y and $f^{-1}(Y - A)$ is closed in X. Since

$$f^{-1}(Y - A) = X - f^{-1}(A). \tag{3}$$

Hence $f^{-1}(A)$ is a open subset of X and f is a continuous function.

● PROBLEM 12-7

Let

$$f: X \rightarrow Y$$

$$g: Y \rightarrow Z \tag{1}$$

be continuous functions.

Show that the composition function

$$g \circ f: X \rightarrow Z \tag{2}$$

is also continuous.

SOLUTION:

Let A represent an open subset of Z. Then $g^{-1}(A)$ is an open subset of Y because g is continuous.

Also $f^{-1}[g^{-1}(A)]$ is an open subset of X, since f is a continuous function.

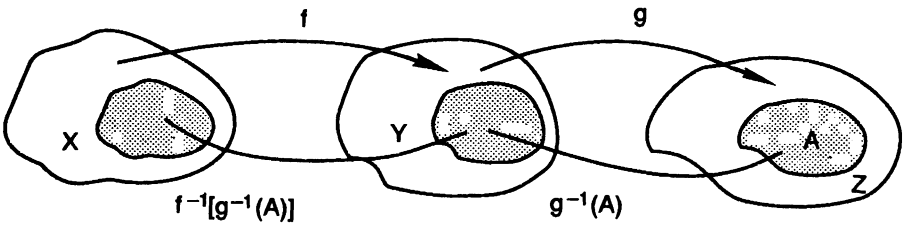

FIGURE 1

But

$$(g \circ f)^{-1}(A) = f^{-1}[g^{-1}(A)]. \tag{3}$$

Thus $(g \circ f)^{-1}(A)$ is an open set in X whenever A is an open set in Z. Hence, $g \circ f$ is a continuous function.

● PROBLEM 12-8

Let

$$f: (X, T_X) \rightarrow (Y, T_Y). \tag{1}$$

Prove that

$$(f \text{ continuous}) \Leftrightarrow \begin{pmatrix} \forall f(x) \in Y \quad \forall V \ni f(x) \\ \exists U \ni x \text{ such that } f(U) \subset V \end{pmatrix}$$

SOLUTION:

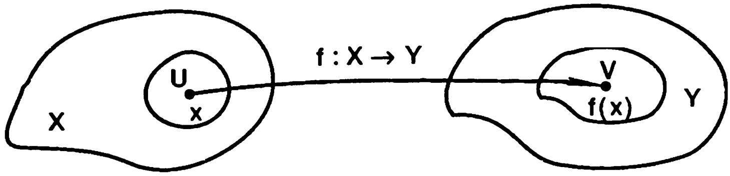

FIGURE 1

$\Rightarrow$ Suppose f is continuous. Take any $f(x) \in Y$ and any neighborhood V of $f(x)$. Then

$$f(x) \in V \text{ and } V \in T_Y. \tag{2}$$

Since f is continuous $f^{-1}(V) \in T_X$.

Also $x \in f^{-1}(V)$. Hence $U = f^{-1}(V)$.

$\Leftarrow$ Let W represent any open subset of Y. Suppose $x \in f^{-1}(W)$, then $f(x) \in W$. Thus W is a neighborhood of $f(x)$. An open set $U \subset X$ exists, such that $x \in U$ and

$$f(U) \subset W. \tag{3}$$

But

$$U \subset f^{-1}(W). \tag{4}$$

Hence, for each $x \in f^{-1}(W)$

$$x \in U \subset f^{-1}(W). \tag{5}$$

where $U \in T_X$.

Therefore $f^{-1}(W)$ is the union of open subsets of X and thus, an open subset.

● PROBLEM 12-9

Let

$$f: (X, T_X) \to (Y, T_Y) \tag{1}$$

and let $B = \{B_\alpha\}$ denote a basis for T_Y. Function f is continuous if and only if for each $B_\alpha \in B, f^{-1}(B_\alpha)$ is an open subset of X.

SOLUTION:

$\Rightarrow$ Suppose f is continuous. Since each $B_\alpha \in B$ is an open subset of Y,

$$f^{-1}(B_\alpha) \in T_X. \tag{2}$$

$\Leftarrow$ Suppose for each $B_\alpha \in B, f^{-1}(B_\alpha)$ is an open set in X. Let $A \subset Y$ denote an open subset in Y. Since B is the basis

$$A = \bigcup_\alpha B_\alpha. \tag{3}$$

Then

$$f^{-1}(A) = f^{-1}(\bigcup_\alpha B_\alpha) = \bigcup_\alpha f^{-1}(B_\alpha). \tag{4}$$

Each $f^{-1}(B_\alpha)$ is an open set. Hence, $f^{-1}(A)$ is an open set as a union of open sets. Function f is continuous.

● PROBLEM 12-10

Let

$$f : (X, T_X) \to (Y, T_Y) \tag{1}$$

and let $S = \{S_\alpha\}$ denote a subbasis for the topology T_Y.

Prove that f is continuous if and only if the inverse of every $S_\alpha \in S$ is an open subset of X.

SOLUTION:

If (Y, T_Y) is a topological space, then a class S of open subsets $S \subset T_Y$ is a subbasis for T_Y if and only if finite intersections of members of S form a basis for T_Y.

$\Rightarrow$ Suppose f is continuous. Then the inverse of all open sets are open. Hence, since S_α are open, sets $f^{-1}(S_\alpha) \in T_X$.

$\Leftarrow$ Suppose for every $S_\alpha \in S$ $f^{-1}(S_\alpha) \in T_X$. Let $A \in T_Y$. Since $\{S_\alpha\}$ is a subbasis,

$$A = \bigcup_\alpha (S_{\alpha_1} \cap \ldots \cap S_{\alpha_k}). \tag{2}$$

We have

$$f^{-1}(A) = f^{-1}\left[\bigcup_\alpha (S_{\alpha_1} \cap \ldots \cap S_{\alpha_k})\right] =$$

$$=\bigcup_{\alpha} f^{-1}(S_{\alpha_1} \cap \dots \cap S_{\alpha_k}) =$$

$$=\bigcup_{\alpha} \left[f^{-1}(S_{\alpha_1}) \cap \dots \cap f^{-1}(S_{\alpha_k})\right]. \tag{3}$$

Since all sets $f^{-1}(S_{\alpha_1}), \dots, f^{-1}(S_{\alpha_k})$ are open, so is the set $f^{-1}(A)$. Hence, function f is continuous.

• PROBLEM 12-11

Let $\{T_\alpha\}$ represent a family of topologies on X. Suppose

$$f: X \to Y \tag{1}$$

is continuous with respect to each topology T_α. Show that f is continuous with respect to the topology

$$T = \bigcap_{\alpha} T_\alpha. \tag{2}$$

SOLUTION:

First note that intersection of any family of topologies is a topology.

Let A denote an open subset of Y, $A \subset Y$. Consider set $f^{-1}(A)$. Since f is continuous with respect to each topology T_α, the set $f^{-1}(A)$ belongs to each T_α, $f^{-1}(A) \in T_\alpha$. Hence

$$f^{-1}(A) \in \bigcap_{\alpha} T_\alpha = T. \tag{3}$$

Therefore f is continuous with respect to T.

• PROBLEM 12-12

Prove this theorem

$$(f: X \to Y \text{ is continuous}) \Leftrightarrow (\forall\, A \subset X : f(\overline{A}) \subset \overline{f(A)}).$$

SOLUTION:

$\Rightarrow$ Suppose $f: X \to Y$ is continuous. Since

$$f(A) \subset \overline{f(A)} \tag{1}$$

we obtain

$$A \subset f^{-1}[f(A)] \subset f^{-1}[\overline{f(A)}]. \tag{2}$$

Set $\overline{f(A)}$ is closed and since f is continuous $f^{-1}[\overline{f(A)}]$ is closed as well

$$f^{-1}[\overline{f(A)}] = \overline{f^{-1}[\overline{f(A)}]}. \tag{3}$$

Hence, from (2) and (3), we find

$$A \subset \overline{A} \subset f^{-1}[\overline{f(A)}]. \tag{4}$$

Thus

$$f(\overline{A}) \subset f[f^{-1}[\overline{f(A)}] = \overline{f(A)}. \tag{5}$$

$\Leftarrow$ Suppose for any $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$. Let B denote any closed subset of Y, $B \subset Y$ and let

$$f^{-1}(B) = A. \tag{6}$$

Then

$$f(\overline{A}) = f[\overline{f^{-1}(B)}] \subset \overline{f[f^{-1}(B)]} = \overline{B} = B. \tag{7}$$

Hence

$$\overline{A} \subset f^{-1}[f(\overline{A})] \subset f^{-1}(B) = A \tag{8}$$

Since $A \subset \overline{A}$, we obtain from (8)

$$A = \overline{A}. \tag{9}$$

Thus, the inverse image of any closed subset of Y is a closed subset of X. Therefore, $f: X \to Y$ is continuous, by Problem 12–6.

● PROBLEM 12–13

Let

$$f: (X, T_X) \to (Y, T_Y) \tag{1}$$

represent a continuous function. Show that

$$f_A: (A, T_A) \to (Y, T_Y) \tag{2}$$

where $f_A = f / A$ is the restriction of f to A, is also continuous.

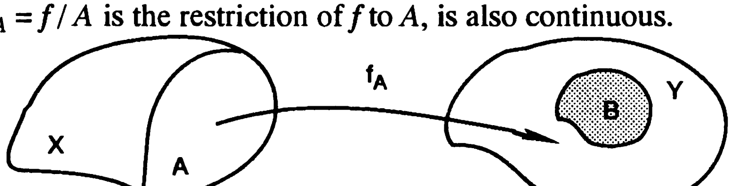

FIGURE 1

SOLUTION:

Note that, for any $B \subset Y$

$$f_A^{-1}(B) = f^{-1}(B) \cap A. \tag{3}$$

Suppose B is an open subset of Y, $B \in T_Y$. Since f is continuous

$$f^{-1}(B) \in T_X. \tag{4}$$

By definition of the induced topology

$$f^{-1}(B) \cap A \in T_A. \tag{5}$$

Hence

$$f^{-1}(B) \cap A = f_A^{-1}(B) \in T_A. \tag{6}$$

and so f_A is continuous.

● PROBLEM 12–14

Prove the following:

THEOREM

If $f: (X, T_X) \to (Y, T_Y)$ is a function and $X = A \cup B$ and $f|A$ and $f|B$ are both continuous (where A and B are topological subspaces of X), then if A and B are both closed or both open, f is continuous.

■

SOLUTION:

If A and B are not both closed or both open, then f does not have to be continuous.

For example, let

$$A = (0, 1) \qquad B = [1, 2) \qquad X = (0, 2)$$

and let $f|A = 0$ and $f|B = 1$. Then both $f|A$ and $f|B$ are continuous, but f is not continuous.

Suppose both A and B are closed. Let $D \subset Y$ denote any closed subset of Y. We will show that if $D \in T_Y$ then

$$f^{-1}(D) \in T_X. \tag{1}$$

$(f|A)^{-1}(D)$ is closed in A and $(f|B)^{-1}(D)$ is closed in B since both $f|A$ and $f|B$ are continuous. Since A and B are closed $(f|A)^{-1}(D)$ and $(f|B)^{-1}(D)$ are closed subsets of X.

From

$$A \cup B = X \tag{2}$$

we obtain

$$f^{-1}(D) = (f\,|\,A)^{-1}(D) \cup (f\,|\,B)^{-1}(D). \tag{3}$$

Equation (3) is the union of two closed subsets of X, hence $f^{-1}(D)$ is a closed subset of X.

Therefore, f is continuous.

● PROBLEM 12–15

Use the theorem of Problem 12–14 to show that the function

$$f : R \to R$$

$$f(x) = \begin{cases} x & \text{if } \ x \le 0 \\ 0 & \text{if } \ x \ge 0 \end{cases} \tag{1}$$

is continuous. R is equipped with the Euclidean topology.

SOLUTION:

Remember that if $f : X \to Y$ and $A \subset X$, then $f\,|\,A$ is the restricted function $f|A : A \to Y$, where $f\,|\,A(x) = f(x)$ for each $x \in A$.

Consider the sets

$$A = \{x \in R : x \le 0\} \quad \text{and} \quad B = \{x \in R : x \ge 0\}. \tag{2}$$

Then

$$R = A \cup B \tag{3}$$

and both A and B are closed.

Both functions $f\,|\,A$ and $f\,|\,B$ are continuous. By the theorem of Problem 12–14, function f is continuous.

● PROBLEM 12–16

Let R^2 (coordinate plane) be a topological space with Euclidean topology. Show that any rotation is continuous.

SOLUTION:

Each point of R^2 can be described by its polar coordinates (r, α). Rotation of R^2 about the origin (0, 0) does not change r and transforms α into

$$\alpha_0 + \alpha$$

where α_0 is the angle of rotation. Thus

$$R_{\alpha_0}(r,\alpha) = (r,\alpha + \alpha_0). \tag{1}$$

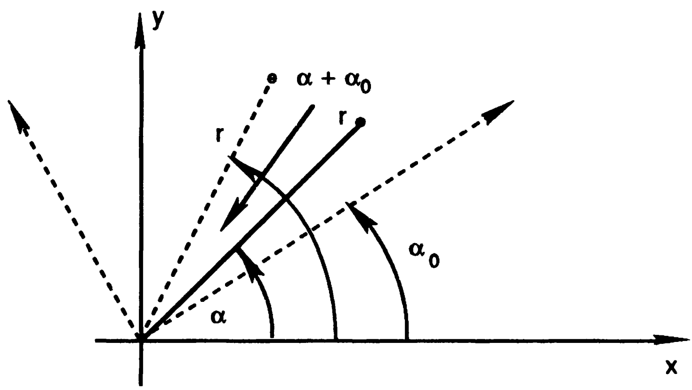

FIGURE 1

Denoting by R_{α_0} rotation through angle α_0, we obtain

$$R_{\alpha_0}^{-1} = R_{-\alpha_0} \tag{2}$$

where $R_{-\alpha_0}$ is the rotation through $-\alpha_0$. Any rotation preserves congruences. Hence, if D is the interior of a square, then $R_{\alpha_0}^{-1}(D)$ is also the interior of a square. The family of interiors of squares forms a basis for R^2 with Euclidean topology.

By Problem 12–9, we conclude that any rotation is continuous.

● PROBLEM 12-17

Show that the projection mappings from the plane R^2 into the line R are continuous with respect to Euclidean topology.

SOLUTION:

Both projections can be defined as follows:

$$P_x, P_y : R^2 \to R$$

$$P_x(x, y) = x$$

$$P_y(x, y) = y \tag{1}$$

Consider $P_y(x, y) = y$. The inverse of any open interval (a, b) is an infinite open strip

$$A = \{(x, y) : a < y < b\}.$$

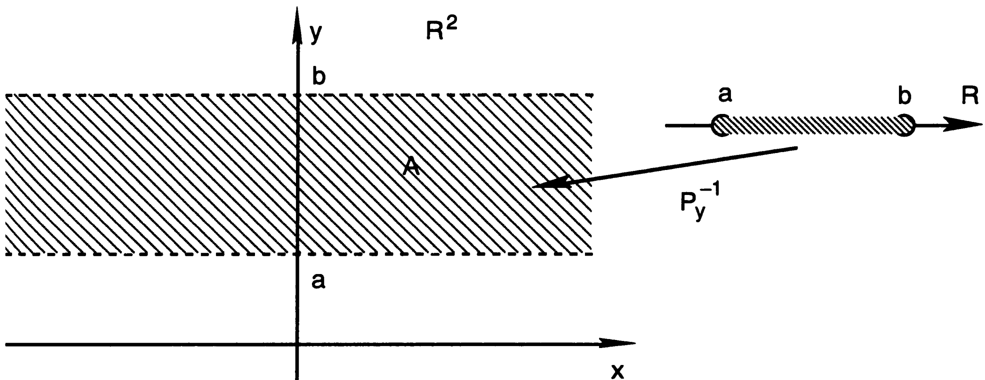

FIGURE 1

The family of open intervals forms a basis for R with the usual topology.

The inverse of any element of the basis is an open set. Hence, by Problem 12–9, the projection P_y is a continuous function. Similarly, we can show that P_x is continuous.

● PROBLEM 12–18

Let (X, T_X) represent a topological space and $\{a\}$ a singleton set, which is an open subset of X, $\{a\} \in T_X$. Show that any function

$$f : X \to Y \tag{1}$$

where (Y, T_Y) is any topological space, is continuous at $a \in X$.

SOLUTION:

DEFINITION OF CONTINUITY AT A POINT

A function $f : X \to Y$ is continuous at $x_0 \in X$ if for each neighborhood $W(f(x_0))$ in Y, a neighborhood $V(x_0)$ in X exists, such that $f(V(x_0)) \subset W(f(x_0))$. ■

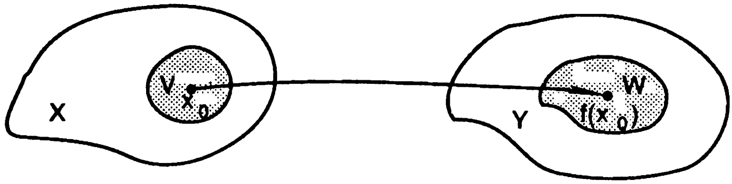

FIGURE 1

Let B represent any open set in Y containing $f(a)$.

$$f(a) \in B. \tag{2}$$

The set $\{a\}$ is an open set in X,

$$\{a\} \in T_X \text{ and } a \in \{a\}. \tag{3}$$

Also, since $f(a) \in B$

$$f(\{a\}) \subset B \tag{4}$$

Thus, f is continuous at $a \in X$.

● PROBLEM 12-19

Consider the topological space (X, T) where

$$X = \{a, b, c, d\} \tag{1}$$

and

$$T = \{\phi, X, \{a\}, \{a, b\}, \{a, b, c\}\}. \tag{2}$$

Show that the function $f: X \to X$, depicted in the diagram, is continuous at $b \in X$ but not continuous at $c \in X$.

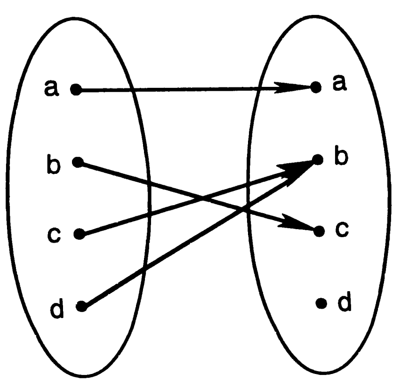

FIGURE 1

SOLUTION:

From the diagram

$$f(b) = c \tag{3}$$

The only open sets containing c are $\{a, b, c\}$ and X. We have

$$f^{-1}(\{a, b, c\}) = X \tag{4}$$

and

$$f^{-1}(X) = X. \tag{5}$$

Thus, the inverse of any open set containing $f(b)$ is an open set containing b, and f is continuous at b.

From the diagram

$$f(c) = b \tag{6}$$

Consider an open set containing $f(c)$

$$\{a, b\} \ni b = f(c), \tag{7}$$

Then

$$f^{-1}(\{a, b\}) = \{a, c, d\} \tag{8}$$

Set $\{a, c, d\}$ does not contain any open set containing c.

Hence, f is not continuous at $c \in X$.

• PROBLEM 12–20

1. Show that a function

$$f: (X, T_X) \to (Y, T_Y) \tag{1}$$

is continuous if and only if it is continuous at every point $a \in X$.

2. Show that if (1) is continuous at $a \in X$, then the restriction of f to A, where $a \in A \subset X$, is also continuous at a.

SOLUTION:

1. Suppose f is continuous. Let $a \in X$ denote any point, and let $B \subset Y$ denote an open subset of Y, such that $f(a) \in B$, $B \in T_Y$. Then $f^{-1}(B) \in T_X$ and $a \in f^{-1}(B)$. Thus, f is continuous at $a \in X$. Now, suppose f is continuous at every point $a \in X$. Let $A \subset Y$ denote an open set.

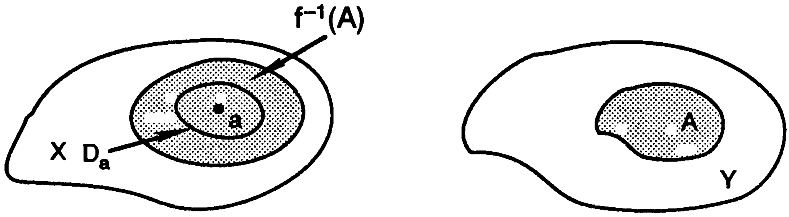

FIGURE 1

For every $a \in f^{-1}(A)$ an open set $D_a \subset X$ exists, such that

$$a \in D_a \subset f^{-1}(A). \tag{2}$$

Thus

$$f^{-1}(A) = \bigcup_a D_a \tag{3}$$

where $a \in f^{-1}(A)$. Set (3) is open as a union of open sets, Hence, f is continuous.

2. Suppose $D \subset Y$ is an open subset containing $f(a)$, see Figure 2.

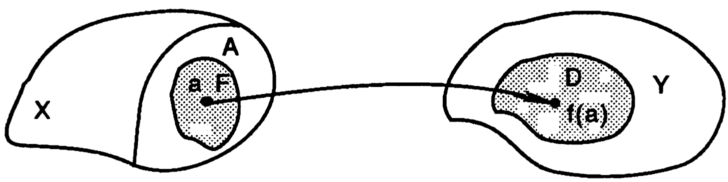

FIGURE 2

Since f is continuous at $a \in X$, there is an open subset F of X, such that

$$a \in F \subset f^{-1}(D). \tag{4}$$

Thus

$$a \in A \cap F \subset A \cap f^{-1}(D) = f_A^{-1}(D) \tag{5}$$

where

$$f_A = f \mid A. \tag{6}$$

But

$$A \cap F \in T_A. \tag{7}$$

Hence f_A is continuous at $a \in X$ with respect to topology T_A.

● PROBLEM 12-21

Show that if a function

$$f: X \to Y \tag{1}$$

is continuous at $a \in X$, then it is sequentially continuous at $a \in X$.

SOLUTION:

DEFINITION OF SEQUENTIAL CONTINUITY

A function $f: X \to Y$ is sequentially continuous at a point $a \in X$ if for every sequence (a_n) in X,

$$(a_n \to a) \Rightarrow (f(a_n) \to f(a)) \tag{2}$$

■

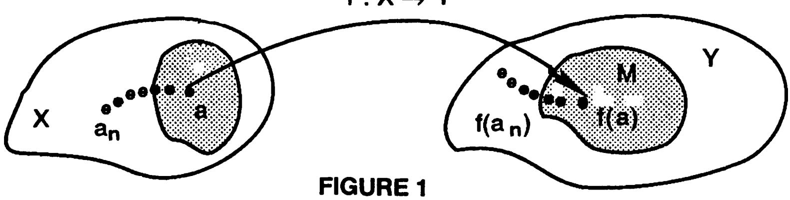

FIGURE 1

Suppose $f: X \to Y$ is continuous at $a \in X$, and let (a_n) denote a sequence convergent to a, $a_n \to a$.

We will show that any neighborhood M of $f(a)$, $f(a) \in M \in T_Y$ contains all but a finite number of the elements of the sequence $(f(a_n))$.

We have

$$f(a) \in M \in T_Y \tag{3}$$

Function f is continuous at $a \in X$. Hence, $f^{-1}(M) = N$ is a neighborhood of a. N contains all but a finite number of elements of (a_n). Then

$$a_n \in N \Rightarrow f(a_n) \in N. \tag{4}$$

Hence, N contains all but a finite number of elements of $f(a_n)$ and

$$f(a_n) \to f(a). \tag{5}$$

● PROBLEM 12-22

Give an example of a neighborhood-finite family in R.

SOLUTION:

DEFINITION OF A NEIGHBORHOOD-FINITE FAMILY

A collection $\{A_\omega : \omega \in \Omega\}$ of sets in a topological space (X, T) is called neighborhood-finite if each point of X has a neighborhood $N_0(x)$, such that

$$N_0(x) \cap A_\omega \neq \emptyset \tag{1}$$

for at most finitely many indices ω.

∎

Consider the family $\{A_p\}$,

$$A_p = [\, p, p + 1\} \tag{2}$$

where p is an integer.

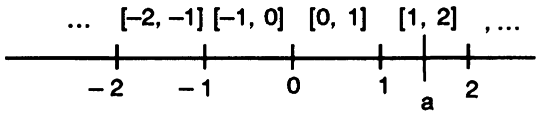

FIGURE 1

Let $a \in R$ denote any point in R. Then, we can always find a sufficiently small neighborhood U of $a \in R$, which intersects with only one element of $\{A_p\}$ (if a is not an integer). If a is an integer, then a sufficiently small neighborhood U of a would intersect with two members of $\{A_p\}$.

Note that a family $\{A_\omega\}$ may be neighborhood-finite, even though each A_α intersects infinitely many other A_β. For example in R_+, we define $\{A_n\}$

$$A_n = \{x : X > n\}. \tag{3}$$

● PROBLEM 12-23

Here is a useful theorem concerning coverings of the space.

THEOREM

Let $\{A_\omega : \omega \in \Omega\}$ represent a family of sets that forms a covering of the space X, that is, $X = \bigcup_\omega A_\omega$. Assume that one of these two conditions holds:

1. all sets A_ω are open, or

2. all sets A_ω are closed, and form an neighborhood-finite family. Then

$$\begin{pmatrix} B \subset X \text{ is open} \\ \\ \text{(or closed)} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \text{all } B \cap A_\omega \text{ are open} \\ \\ \text{(or closed) is the subspace } A_\omega \end{pmatrix}.$$

■

Prove this theorem for the case when condition 2 is true.

SOLUTION:

$\Rightarrow$ This follows immediately from the definition of induced topology.

$\Leftarrow$ Suppose each $B \cap A_\omega$ is closed in the closed A_ω.

The case when each $B \cap A_\omega$ is open in the closed A_ω is similar. Since a subspace of a subspace is a subspace of the entire space X *and each* $B \cap A_\omega$ is closed in the closed A_ω it follows that $B \cap A_\omega$ is closed in X.

$$B \cap A_\omega = \overline{B \cap A_\omega}. \tag{1}$$

Since $\{A_\omega\}$ is neighborhood-finite, so is $\{B \cap A_\omega\}$. Therefore (remember, $\{A_\omega\}$ is a covering)

$$B = \bigcup_\omega \{B \cap A_\omega\} \tag{2}$$

is closed in X.

Here we applied the following:

THEOREM

If $\{A_\omega : \omega \in \Omega\}$ is an neighborhood-finite family in X, then for each $\Lambda \subset \Omega$, the set

$$\bigcup_\lambda \{\overline{A}_\lambda : \lambda \in \Lambda\} \tag{3}$$

is closed in X.

● PROBLEM 12-24

The situation in which a continuous function is defined piecewise appears in analysis. For example, a function which is continuous on a segment $[0, n] \subset R$ is defined for each $[k, k + 1]$ separately in such a way, that the adjacent functions agree on the common end points of the segments $[0, 1]$, $[1, 2]$, ..., $[n - 1, n]$.

Partial definitions of functions are formulated in this theorem.

THEOREM

Let (X, T) represent a topological space and $\{A_\omega : \omega \in \Omega\}$ its cover-

ing. One of two conditions is true:

1. all sets A_ω are open, or
2. all sets A_ω are closed and form a neighborhood-finite family.

For each $\omega \in \Omega$ function

$$f_\omega : A_\omega \to Y \tag{1}$$

is continuous and, such that

$$f_\alpha \mid A_\alpha \cap A_\beta = f_\beta \mid A_\alpha \cap A_\beta \tag{2}$$

for each $\alpha, \beta \in \Omega$.

Then a unique continuous function exists

$$f : X \to Y \tag{3}$$

such that for each $\omega \in \Omega$

$$f \mid A_\omega = f_\omega. \tag{4}$$

■

Prove this theorem.

SOLUTION:

For each $x \in X$, we define

$$f(x) = f_\alpha(x) \tag{5}$$

where $\alpha \in \Omega$ is any index, such that

$$x \in A_\alpha. \tag{6}$$

The definition is unique, because if $x \in A_\alpha$ and $x \in A_\beta$ then

$$f(\text{x}) = f_\alpha(x) = f_\beta(x) \tag{7}$$

Since $f_\alpha \mid A_\alpha \cap A_\beta = f_\beta \mid A_\alpha \cap A_\beta$. Hence (5) defines a function which is unique.

Function f is continuous. Let $U \subset Y$ be open. Then

$$f^{-1}(U) \cap A_\omega = f_\omega^{-1}(U) \tag{8}$$

is open in A_ω for each ω.

Thus, by Problem 12–23, $f^{-1}(U)$ is open in X and f is continuous.

● PROBLEM 12–25

Show that the function

$$f: R \to R$$

such that

$$f(x) = x^2 \tag{1}$$

is not open.

SOLUTION:

Observe that if f is a continuous function, then the inverse image of every open set is an open set and the inverse image of every closed set is a closed set. It will be useful to define the following group of functions.

DEFINITION OF AN OPEN (CLOSED) FUNCTION

A function

$$f: X \to Y \tag{2}$$

is called an open (closed) function, if the image of every open (closed) set is open (closed).

■

Consider an open interval $(-1, 1)$. Function $f(x) = x^2$ maps $(-1, 1)$ into

$$f((-1, 1)) = [0, 1) \tag{3}$$

which is not an open set. Hence, f defined by (1) is not open.

● PROBLEM 12–26

Let $f: X \to Y$ denote a function and B represent a basis for a topological space (X, T).

Show that, if for every $B_\alpha \in B$ $f(B_\alpha)$ is open in Y, then f is an open function.

SOLUTION:

Suppose A is an open subset of X. Then

$$A = \bigcup_\alpha B_\alpha, \; B_\alpha \in B \tag{1}$$

since B is the basis for X. Hence

$$f(A) = f(\bigcup_\alpha B_\alpha) = \bigcup_\alpha f(B_\alpha). \tag{2}$$

By hypothesis, each $f(B_\alpha)$ is open in Y. Therefore $f(A)$ is open, as a union of open sets, and f is an open function.

As a matter of fact two properties are equivalent:

1. f is an open function.

2. f maps each member of a basis for X to an open set in Y.

• PROBLEM 12-27

Let

$$p : R^2 \rightarrow R \tag{1}$$

represent the projection mapping of R^2 into the x-axis, i.e.,

$$p(x, y) = x. \tag{2}$$

1. Show that p is an open function.

2. Show that p is not a closed function.

SOLUTION:

1. The family of all open discs in R^2 forms a basis for R^2. We assume that R^2 is equipped with the Euclidean topology.

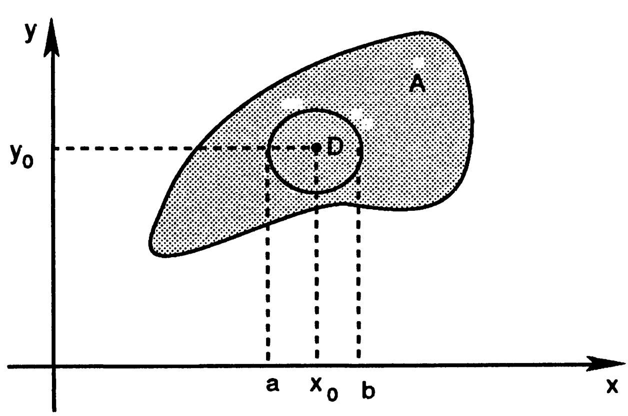

FIGURE 1

Note that the projection of any disc D

$$D = \{(x, y) : (x - x_0)^2 + (y - y_0)^2 < r^2\} \tag{3}$$

is an open interval

$$p(D) = (a, b) \tag{4}$$

where (a, b) is an open interval. According to Problem 12–26, we conclude that p, defined by (2), is an open function.

2. Consider the shaded region shown in Figure 2.

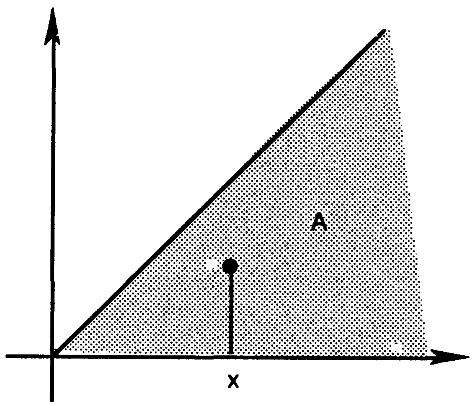

FIGURE 2

$$A = \{(x, y) : x \geq 0, 0 \leq y \leq x\}. \tag{5}$$

Set A is closed, while its projection $p(A) = [0, \infty)$ is not closed. Thus, p is not a closed function.

● PROBLEM 12–28

Prove the following theorem:

THEOREM

Let $f : X \to Y$ represent a closed function and let D denote any subset of Y. Let U denote any open set, such that

$$f^{-1}(D) \subset U \subset X. \tag{1}$$

Then an open V exists, such that $D \subset V$ and $f^{-1}(V) \subset U$.

■

See Figure 1.

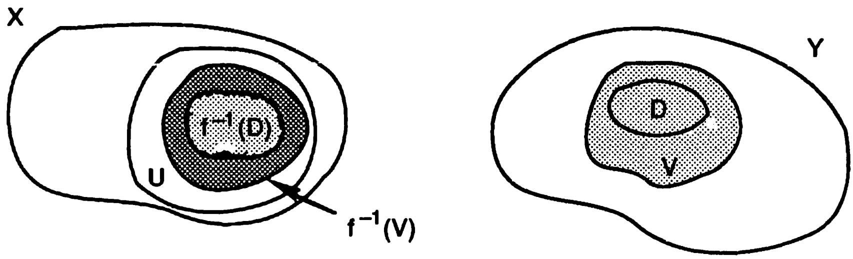

FIGURE 1

SOLUTION:

Let us define

$$V = Y - f(X - U). \tag{2}$$

By hypothesis

$$f^{-1}(D) \subset U. \tag{3}$$

Therefore

$$D \subset V.$$

Indeed, suppose $y \in D$, then $f^{-1}(y) \subset f^{-1}(D)$. Hence, $f^{-1}(y) \subset U$ and $y \in f(U)$. Thus, $y \notin f(X - U)$ and $y \in Y - f(X - U)$. Finally

$$y \in V \tag{4}$$

By hypothesis, f is closed and U is an open subset of X. Then $X - U$ is closed and $f(X - U)$ is closed. Therefore $V = Y - f(X - U)$ is an open subset of Y. Now we prove that $f^{-1}(V) \subset U$.

$$f^{-1}(V) = X - f^{-1}[f(X - U)] \subset X - (X - U) = U \tag{5}$$

A similar theorem exists for open functions.

THEOREM

Let $f : X \to Y$ denote an open function and let D represent any subset of Y. Let W represent any closed set, such that

$$f^{-1}(D) \subset W.$$

Then a closed set P exists, such that $D \subset P$ and

$$f^{-1}(P) \subset W.$$

∎

● PROBLEM 12-29

Prove

$$(f: X \to Y \text{ is a closed function}) \Leftrightarrow (\forall A \subset X: \overline{f(A)} \subset f(\overline{A})).$$

SOLUTION:

$\Rightarrow$ Suppose f is a closed function. Let A represent any subset of X. Then $\overline{A}$ is a closed set and $f(\overline{A})$ is closed.

Since

$$f(a) \subset f(\overline{A}) \tag{1}$$

we have

$$\overline{f(A)} \subset \overline{f(\overline{A})} = f(\overline{A}). \tag{2}$$

$\Leftarrow$ Suppose, for each set $A \subset X$, $\overline{f(A)} \subset f(\overline{A})$.

Let A represent a closed set, $\overline{A} = A$. We have

$$f(A) \subset \overline{f(A)} \subset f(\overline{A}) = f(A). \tag{3}$$

Therefore

$$f(A) = \overline{f(A)} \tag{4}$$

so that set $f(A)$ is closed and function $f: X \to Y$ is closed.

● PROBLEM 12-30

Use the function

$$f(x) = \frac{x}{|x| + 1} \tag{1}$$

to show that the spaces R' and $(-1, +1)$ are homeomorphic.

SOLUTION:

DEFINITION OF HOMEOMORPHISM

A continuous bijective (that is, one-to-one and onto) function $f: X \to Y$, such that $f^{-1}: Y \to X$ is also continuous, is called a homeomorphism and denoted by

$$f: X \cong Y. \tag{2}$$

Two spaces X, Y, denoted by $X \cong Y$, are homeomorphic (or of the same topological type), if there is a homeomorphism $f : X \cong Y$.

■

First of all, observe that

$$f(x) = \frac{x}{|x| + 1}$$

where $f : R \to (-1, +1)$ is one-to-one and onto. Both functions f and f^{-1} are continuous. Hence, f, defined by (1), is a homeomorphism.

This result can be generalized to n-dimensional space

$$f : R^n \to B(0, 1) \tag{3}$$

where $B(0, 1)$ is the unit ball. Then $x = (x_1, x_2, \ldots, x_n)$ and

$$|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

where

$$f(x) = \frac{x}{|x| + 1} \tag{4}$$

Hence, R^n is homeomorphic to its unit ball $B(0, 1)$

$$R^n \cong B(0, 1). \tag{5}$$

● PROBLEM 12-31

Show that the homeomorphism relation is reflexive, symmetric and transitive.

SOLUTION:

Relation is reflexive. For any topological space (X, T), the identity mapping $f(x) = x$

$$f : X \to X \tag{1}$$

is a homeomorphism.

Symmetry. If $X \cong Y$, then a homeomorphism exists, such that

$$f : X \to Y. \tag{2}$$

We define $g = f^{-1}$, where

$$g : Y \to X. \tag{3}$$

Mapping g is a homeomorphism. Hence, $Y \cong X$.

We will show that the homeomorphism relation is transitive.

Suppose $X \cong Y$ and $Y \cong Z$. Then $f: X \to Y$ and $g: Y \to Z$ are homeomorphisms.

Mapping

$$g \circ f: X \to Z \tag{4}$$

where $g \circ f(x) = g[f(x)]$ is a homeomorphism, since both g and f are homeomorphisms.

Relation $\cong$ (X is homeomorphic to Y), defined in any family of topological spaces, is an equivalence relation. Hence, any family of topological spaces can be partitioned into disjoint classes of topologically equivalent spaces.

From now on, we will be more concerned with these classes of topologically equivalent spaces, than with the individual topological spaces.

● PROBLEM 12–32

Show that an area is not a topological property.

SOLUTION:

In Problem 12–31, we defined the classes of topologically equivalent spaces. Now we shall investigate the properties which are common for all members of the same class.

DEFINITION OF TOPOLOGICAL PROPERTIES

A property P of sets is called topological or a topological invariant if, whenever a topological space (X, T) has this property, then every space homeomorphic to (X, T) has property P.

■

That is, property P is common for all members of a class of topologically equivalent spaces.

Consider the function

$$f: R^2 \to R^2 \tag{1}$$

defined by

$$f: (x, y) \to (2x, y), \tag{2}$$

where f is a homeomorphism. It is easy to see that f transforms a unit square into a rectangle, as shown in Figure 1.

Hence, a figure of area one is transformed into a figure of area two.

Area is not a topological property. Similarly, it can be shown that length and volume are not topological properties.

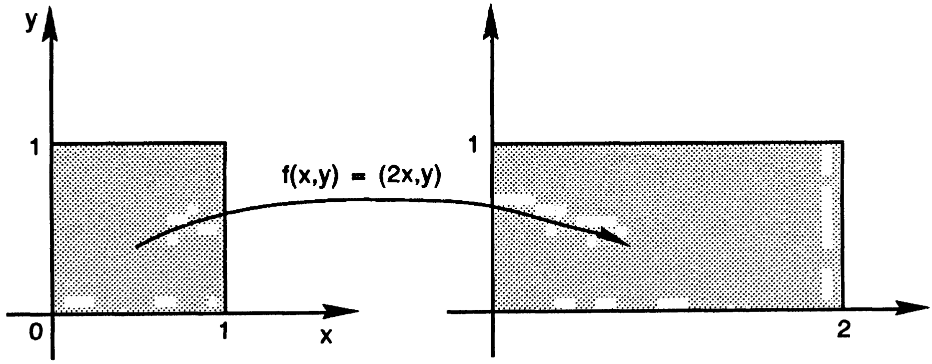

FIGURE 1

● PROBLEM 12-33

Show that each of the following conditions is necessary and sufficient for a one-to-one mapping, f, to be a homeomorphism:

1. $f(\overline{A}) = \overline{f(A)}$ for every $A \subset X$. (1)

2. $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$ for every $B \subset Y$. (2)

SOLUTION:

We shall apply the following:

THEOREM

$$(f \text{ is continuous}) \Leftrightarrow \left(f(\overline{A}) \subset \overline{f(A)} \text{ for every } A \subset X\right) \tag{3}$$

$$(f \text{ is continuous}) \Leftrightarrow \left(\overline{f^{-1}(B)} \subset f^{-1}(\overline{B}) \text{ for every } B \subset Y\right). \tag{4}$$

Suppose f is a homeomorphism, then f is continuous and for every $A \subset X$

$$f(\overline{A}) \subset \overline{f(A)} \tag{5}$$

Also $f^{-1}: Y \to X$ is continuous and by (4), we obtain

$$\overline{f(A)} \subset f(\overline{A}). \tag{6}$$

Note that $(f^{-1})^{-1} = f$. From (5) and (6), we obtain

$$f(\overline{A}) = \overline{f(A)}. \tag{7}$$

Suppose, for every $A \subset X$, $f(\overline{A}) = \overline{f(A)}$. Then

$$f(\overline{A}) \subset \overline{f(A)} \tag{8}$$

and by (3), we conclude that f is continuous.

Also, for every $A \subset X$

$$\overline{f(A)} = f^{-1}[f^{-1}(A)] \subset f^{-1}[f^{-1}(\overline{A})] = f(\overline{A}). \tag{9}$$

Hence, $f^{-1} : Y \rightarrow X$ is continuous and f is a homeomorphism.

Similarly, we show that condition (2) is necessary and sufficient for a one-to-one function to be homeomorphic.

• PROBLEM 12–34

Suppose f maps X onto Y, $f : X \rightarrow Y$, where X is a T_1–space.

Show that a necessary and sufficient condition for f to be a homeomorphism is:

1. $\overline{A} = f^{-1}(\overline{f(A)})$ for every $A \subset X$ or (1)

2. $(x \in \overline{A}) \equiv (f(x) \in \overline{f(A)})$ (2)

SOLUTION:

DEFINITION OF A T_1-SPACE

A topological space (X, T) is called a T_1-space, if each single element set is closed, that is,

$$\overline{\{a\}} = \{a\} \text{ for each } a \in X.$$

■

For example, each metric space is a T_1-space.

We shall prove : for $f : X \rightarrow Y$, where X is a T_1-space

$$(f \text{ is a homeomorphism}) \Leftrightarrow (\overline{A} = f^{-1}(\overline{f(A)}) \text{ for every } A \subset X). \tag{3}$$

⇒ Since f is a homeomorphism, it is thus one-to-one. From Problem 12–33, equation (1),

$$f(\overline{A}) = \overline{f(A)} \tag{4}$$

or

$$f^{-1}[f(\overline{A})] = \overline{A} = f^{-1}[\overline{f(A)}]. \tag{5}$$

$\Leftarrow$ We have to show that f is one-to-one. Suppose $f(a) = f(b)$. Then

$$\overline{\{a\}} = f^{-1}[\overline{f(a)}] = f^{-1}[\overline{f(b)}] = \overline{\{b\}}. \tag{6}$$

Since (X, T) is a T_1-space,

$$a = b. \tag{7}$$

Thus, f is one-to-one. Then from (1),

$$f(\overline{A}) = \overline{f(A)} \tag{8}$$

for every $A \subset X$.

According to Problem 12–33, f is a homeomorphism.

Conditions (1) and (2) are equivalent. From (2), we conclude that every property, expressed in terms of the operation $\overline{A}$ and of operations of set theory and of logics, is topological. We can briefly say that if a point $a \in X$ (or a set, family of sets, etc.) has a given property with respect to the space (X, T), then $f(a)$ has the same property with respect to Y, provided that $f: X \to Y$ is a homeomorphism.

● PROBLEM 12–35

1. Show that two closed intervals, $[a, b]$ and $[c, d]$, are homeomorphic.

2. Show that an open interval $(-1, 1)$ and the real line are homeomorphic.

3. Show that the surface of the sphere with one point removed is homeomorphic to the plane.

SOLUTION:

1. Suppose $a < b$ and $c < d$. Define $f(x)$ by

$$f(x) = \frac{d-c}{b-a}x + \frac{bc-ad}{b-a} \tag{1}$$

f is a homeomorphism, which maps the first interval onto the second. Hence, the two closed intervals are homeomorphic. To show that, one can also use the drawing shown in Figure 1.

2. Function $f: (-1, 1) \to R$

$$f(x) = \tan\frac{\pi x}{2} \tag{2}$$

is a homeomorphism. Hence, $(-1, 1)$ and R are homeomorphic.

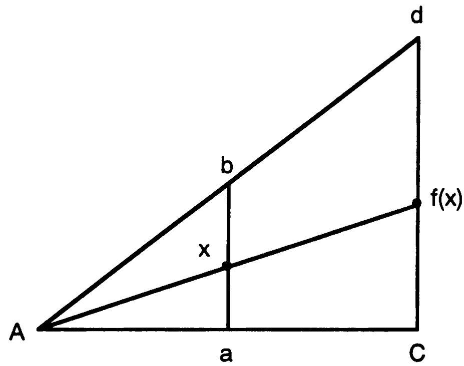

FIGURE 1

3. Consider the surface

$$x^2 + y^2 + (z - 1)^2 = 1 \tag{3}$$

with the point (0, 0, 2) removed. (See Figure 2).

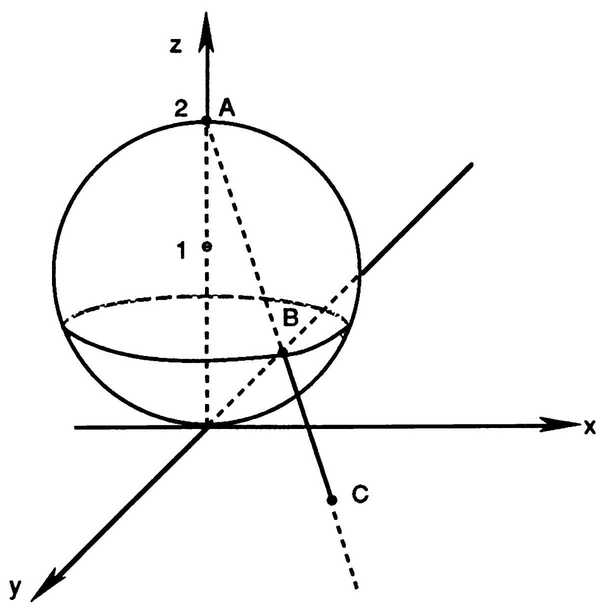

FIGURE 2

Draw a line from A through the point B on the surface of the sphere and through point C on the plane xy.

Each point B on the surface of the sphere is mapped on the plane $z = 0$. This mapping is one-to-one, onto and continuous; the inverse mapping also has the same properties. Thus, a sphere with one point removed and a plane are homeomorphic.

THEOREM

Let $f : X \to Y$ denote a bijection. Then the following properties are equivalent:

1. f is a homeomorphism.
2. $f(\overline{A}) = \overline{f(A)}$ for every $A \subset X$.
3. f is continuous and open.
4. f is continuous and closed.

■

Prove it.

SOLUTION:

1. $\Leftrightarrow$ 2. This was proved in Problem 12-33.

1. $\Leftrightarrow$ 3. f is a homeomorphism, hence $f^{-1} : Y \to X$ is continuous. Therefore, for each open $A \subset X$,

$$f^{-1}[f^{-1}(A)] = f(A) \tag{1}$$

and $f(A)$ is open in Y.

Suppose f is continuous and open, then the image of each open set in X is an open set in Y. Hence, f^{-1} is continuous and f is a homeomorphism.

3. $\Leftrightarrow$ 4. Note, that if $f : X \to Y$ is bijective, then the conditions that f is open and f is closed, are equivalent. Suppose f is open and $A \subset X$ is closed. Then set B is open

$$A = X - \mathrm{B} \tag{2}$$

and

$$f(A) = f(X - B) = f(X) - f(B) = Y - f(B) \tag{3}$$

Hence, since $f(B)$ is open, the set $f(A)$ is closed. Thus, f is closed.

● PROBLEM 12-37

The following theorem helps to determine a given function as being a homeomorphism:

THEOREM 1

Let $f : X \to Y$ and $g : Y \to X$ be continuous and, such that

$$f \circ g = 1_Y \qquad (1)$$

and

$$g \circ f = 1_X \qquad (2)$$

then f is a homeomorphism and

$$f^{-1} = g. \qquad (3)$$

■

Prove this theorem.

SOLUTION:

First let us prove this.

THEOREM 2

Let $f : X \to Y$ and $g : Y \to X$, such that

$$g \circ f = 1_X \qquad (4)$$

then f is one-to-one and g is onto.

■

The map of f is one-to-one, since

$$(f(x) = f(y)) \Rightarrow (x = g \circ f(x) = g \circ f(y) = y). \qquad (5)$$

Function g is onto, since for any $x \in X$,

$$x = g[f(x)]. \qquad (6)$$

By applying Theorem 2 and conditions (1) and (2), we conclude that both f and g are bijective. In addition to that, $f^{-1} = g$. Thus, f and f^{-1} are continuous and f is a homeomorphism.

● PROBLEM 12-38

Let

$$f : (X, T_X) \to (Y, T_Y) \qquad (1)$$

denote a homeomorphism and let (A, T_A) represent a subspace of (X, T_X). Let

$$f_A : (A, T_A) \to (B, T_B) \tag{2}$$

be a restriction of f to A, $f_A = f \mid A$ and let

$$B = f(A)$$

and let T_B be a topology induced on B.

Show that f_A is a homeomorphism.

SOLUTION:

Since f is a bijection (that is, one-to-one and onto), $f_A : A \to B$ is also a bijection. The restriction of a continuous function is also a continuous function. Hence, f_A is continuous.

Let $U \subset A$ be T_A open, then

$$U = A \cap V \tag{3}$$

where $V \in T_X$. Since f is one-to-one,

$$f(A \cap V) = f(A) \cap f(V) \tag{4}$$

Thus,

$$f_A(U) = f(U) = f(A) \cap f(V) = B \cap f(V) \tag{5}$$

Since f is open and $V \in T_X$, $f(V) \in T_Y$. Therefore,

$$B \cap f(V) \in T_B \tag{6}$$

and so, f_A is open.

● PROBLEM 12–39

Show that the subsets X and Y of the plane R^2 with the Euclidean topology, shown in Figure 1, are not homeomorphic. The topologies of X and Y are the usual induced topologies.

$$X = \{x : d(x, a_1) = 1 \text{ or } d(x, a_2) = 1\} \tag{1}$$

$$Y = \{y : d(y, a_3) = 1\} \tag{2}$$

$$a_1 = (0, 1) \quad a_2 = (0, -1) \quad a_3 = (3, 0).$$

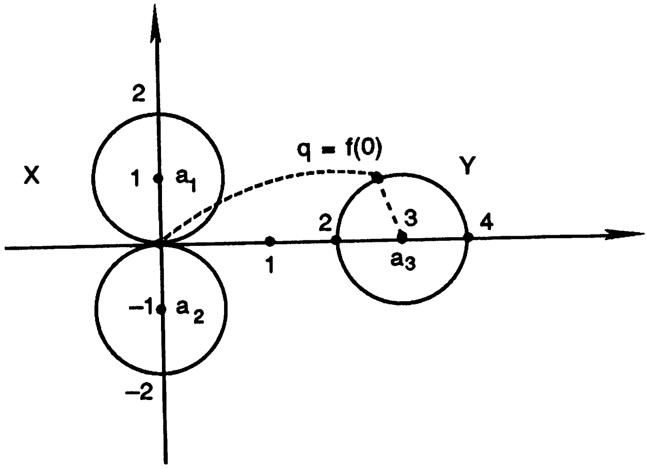

FIGURE 1

SOLUTION:

Suppose a homeomorphism

$$f : X \to Y \tag{3}$$

exists. Let us denote

$$f(0) = q \tag{4}$$

$$X_1 = X - \{0\}$$

$$Y_1 = Y - \{q\}. \tag{5}$$

By Problem 12–38, we conclude that

$$f_1 : X_1 \to Y_1 \tag{6}$$

is also a homeomorphism, with respect to the induced topologies.

Set Y_1 is connected by setting

$$q = (3 + \cos \beta_0, \sin \beta_0), \tag{7}$$

we define

$$g : (0, 2\pi) \to Y_1 \tag{8}$$

by

$$g(\beta) = (3 + \cos(\beta_0 + \beta), \sin(\beta_0 + \beta)). \tag{9}$$

Function g is homeomorphic interval $(0, 2\pi)$ is connected, and thus, Y_1 is connected.

To obtain a contradiction, we will show that X_1 is not connected.

The sets

$$A = \{(x, y) : x > 0\}$$

$$B = \{(x, y) : x < 0\} \tag{10}$$

are both open in R^2. Hence,

$$A_1 = X_1 \cap A$$

$$B_1 = X_1 \cap B \tag{11}$$

are open subsets of X_1, such that

$$A_1, B_1 = \phi, \quad A_1 \cap B_1 = \phi, \quad A_1 \cup B_1 = X_1 \tag{12}$$

space X_1 is not connected. Hence, X_1 and Y_1 are not homeomorphic, because connectedness is a topological property.

• PROBLEM 12-40

A topological space, (X, T), is given, along with an equivalence relation R. Construct the identification topology on ${}^X/_R$ (sometimes called the quotient topology).

SOLUTION:

If X is a set and R is an equivalence relation on X, then R determines a partition of X into equivalence classes.

Two elements, $x, y \in X$, belong to the same class, if and only if, $x R y$ (x is in R-relation to y). The set of equivalence classes is denoted by ${}^X/_R$.

Let us define a mapping

$$f : X \to {}^X/_R \tag{1}$$

by

$$f(x) = [x] \tag{2}$$

where $[x]$ is the equivalence class, such that $x \in [x]$.

f is called the identification mapping (or quotient mapping). See Figure 1.

Let us introduce a topology on ${}^X/_R$.

A subset U of ${}^X/_R$ is open, if and only if, $f^{-1}(U)$ is open in X. In Problem 12-41, we will show that this is, indeed, a topology. This topology is called the identification topology or quotient topology.

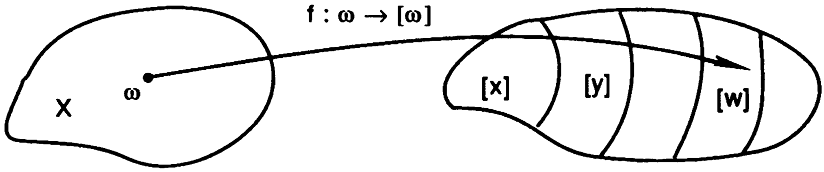

FIGURE 1

● PROBLEM 12–41

Prove that the identification topology, defined in Problem 12–40, is indeed, topology.

SOLUTION:

Let us denote the collection of all open sets of ${}^{X}/_{R}$ by T_R.

1. $$f^{-1}(\phi) = \phi \text{ and } f^{-1}({}^{X}/_{R}) = X \tag{1}$$

(X, T) is a topological space, hence, $\phi, X \in T$. Thus,

$$\phi \in T_R \text{ and } {}^{X}/_{R} \in T_R \tag{2}$$

2. Let $E, F \in T_R$, then $f^{-1}(E)$ and $f^{-1}(F)$ are open sets in X. Thus,

$$f^{-1}(E) \cap f^{-1}(F) \in T. \tag{3}$$

But

$$f^{-1}(E) \cap f^{-1}(F) = f^{-1}(E \cap F) \in T. \tag{4}$$

Hence, $E \cap F$ is an open set in ${}^{X}/_{R}$

$$E \cap F \in T_R \tag{5}$$

3. Let $\{E_\alpha\}$ represent a family of open sets in ${}^{X}/_{R}$. Then each set

$$f^{-1}(E_\alpha) \in T \tag{6}$$

and

$$\bigcup_\alpha f^{-1}(E_\alpha) \in T. \tag{7}$$

We have

$$\bigcup_\alpha f^{-1}(E_\alpha) = f^{-1}(\bigcup_\alpha E_\alpha) \in T \tag{8}$$

and $\bigcup_{\alpha} E_{\alpha}$ is an open subset of $X/_R$.

The collection of open sets of $X/_R$ forms a topology on $X/_R$.

● PROBLEM 12-42

The closed interval [0, 1] is equipped with the absolute value topology. An equivalence relation is defined by:

1. 0 is equivalent to 1

2. every other element of the segment [0, 1] is equivalent only to itself.

The equivalence classes are $\{0, 1\}$ and $\{x\}$ for $0 < x < 1$.

Show that the identification space defined is homeomorphic to a circle.

SOLUTION:

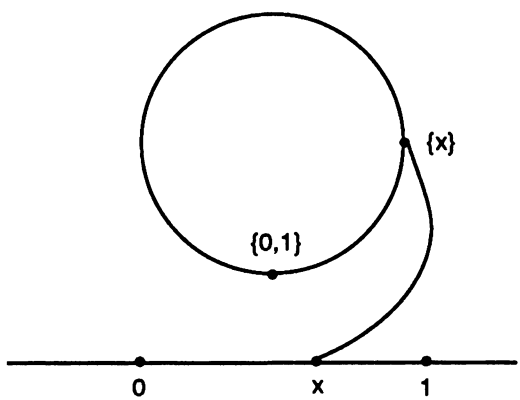

FIGURE 1

The endpoints 0 and 1 of the segment [0, 1] become a single point {0, 1} of the new topological space. Hence, we wrap the segment [0, 1] around a circle of a radius of $^1/_{2\pi}$ and obtain a continuous function from [0, 1] onto a circle. The inverse function is also continuous.

The identification space, obtained from [0, 1], is homeomorphic to a circle.

CHAPTER 13

SEPARATION AXIOMS

Let $X = \{0, 1\}$. Show that X, with the indiscrete topology, is not a T_0–space. Give an example of topology T, for which (X, T) is a T_0–space.

SOLUTION:

DEFINITION OF T_0–SPACE

The space (X, T) is said to be a T_0–space, if for any two distinct $a, b \in X$, there is a neighborhood of at least one, which does not contain the other.

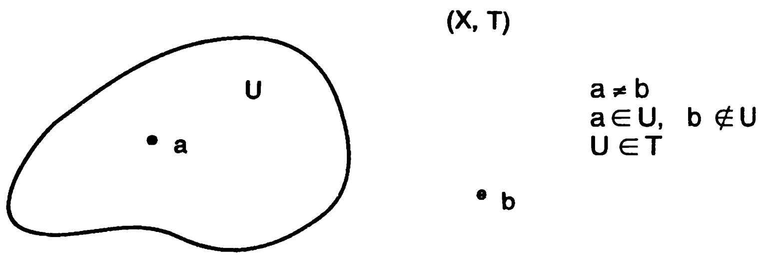

FIGURE 1

■

With the indiscrete topology $T = \{\phi, X\}$ one cannot separate 0 from 1 or 1 from 0. Consider the set $X = \{0, 1\}$ with topology $T = \{\phi, X, \{0\}\}$. Then, for the two distinct points 0 and 1, an open set $\{0\}$ exists, such that

$$0 \in \{0\} \text{ but } 1 \notin \{0\}.$$

Hence, (X, T) is a T_0–space. Note that each metric space (X, d) is a T_0–space.

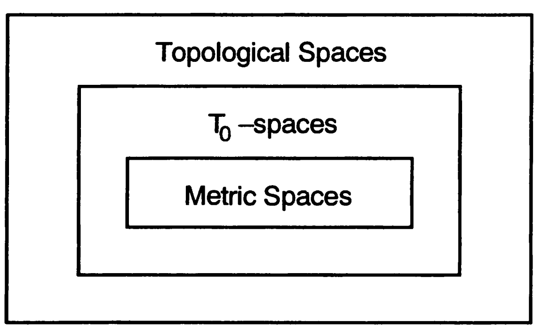

FIGURE 2

● PROBLEM 13–2

Explain why, in general, a pseudometric space is not T_0.

SOLUTION:

A pseudometric on X is a function $D : X \times X \to R_+$, such that

1. $D(x, y) = D(y, x)$

2. $D(x, y) \leq D(x, z) + D(z, y)$

for all $x, y, z, \in X$.

Thus, it is possible that for two distinct points $a, b \in X$,

$$D(a, b) = 0.$$

Any neighborhood of a contains b and any neighborhood of b contains a.

Hence, a pseudometric space is not a T_0–space. Points $a, b \in X$ are distinct, but such that, $D(a, b) = 0$ cannot be separated.

● PROBLEM 13–3

Show that every metric space is a T_1–space.

SOLUTION:

We shall start with the

DEFINITION OF T_1–SPACE

A topological space (X, T) is called a T_1–space, if every single element set is closed, that is,

$$\forall\, a \in X \quad \{a\} = \overline{\{a\}}. \tag{1}$$

■

Note, that in a metric space (X, d), the condition for a set A to be closed, can be expressed by the implication

$$\begin{pmatrix} \lim\limits_{n\to\infty} x_n = x \\ x_n \in A \end{pmatrix} \Rightarrow (x \in \overline{A}).$$

Since lim $a = a$, we have

$$\overline{\{a\}} = \{a\}.$$

Thus, every metric space is a T_1–space (see Figure 1).

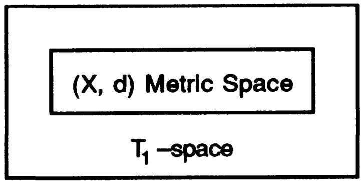

FIGURE 1

There are topological spaces which are not T_1–spaces. For example, space X, containing two points $X = \{a, b\}$ with topology $T = \{\phi, X\}$, is not a T_1–space.

We showed before, that every metric space can be regarded as a topological space. Figure 2 illustrates the results.

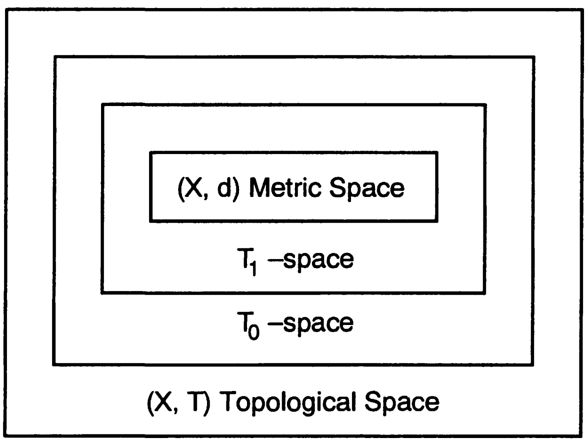

FIGURE 2

From the definitions, it is clear that any T_1–space is also a T_0–space.

● PROBLEM 13-4

Show that a topological space (X, T) is a T_1–space iff for any pair of distinct points $a, b \in X$, the open sets $G, H \in T$ exist, such that

$$a \in G, \quad b \notin G \quad \text{and} \quad b \in H, \quad a \notin H.$$

See Figure 1.

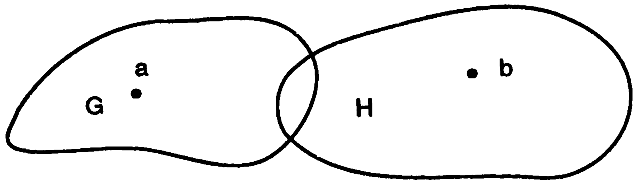

FIGURE 1

SOLUTION:

Suppose (X, T) is a T_1–space. Then, for any $x \in X$, $\{x\}$ is a closed set. Let $a, b \in X$ and $a \neq b$. The sets $X - \{a\}$ and $X - \{b\}$ are open, and

$$a \in X - \{b\} \quad \text{and} \quad b \notin X - \{b\}$$

$$b \in X - \{a\} \quad \text{and} \quad a \notin X - \{a\}.$$

Conversely, suppose $x \in X$. We shall show that $\{x\}$ is closed, i.e., $X - \{x\}$ is open. Let $y \in X - \{x\}$, then $y \neq x$ and an open set H_y exists, such that

$$y \in H_y \quad \text{and} \quad x \in H_y.$$

Thus,

$$y \in H_y \subset X - \{x\} \text{ and } X - \{x\} = \bigcup_{y \neq x} H_y.$$

Since all H_y are open sets, $X - \{x\}$ is open and $\{x\}$ is closed, $\{x\} = \overline{\{x\}}$.

● PROBLEM 13–5

Let X represent a finite set. Prove that the only topology on X, which makes X into a T_1–space, is the discrete topology.

SOLUTION:

Let X represent any finite set and T a topology on X, such that (X, T) is a T_1–space.

A space (X, T) is t_1, iff every one-point subset of X is closed. Since the union of two closed subsets is a closed subset, we conclude that all subsets of X are closed.

Therefore, all subsets of X are open and T is the discrete topology.

• PROBLEM 13-6

Let (X, T) denote a T_1–space. Show that the following conditions are equivalent:

1. $a \in X$ is an accumulation point of A.

2. every open set containing a contains an infinite number of points of A.

SOLUTION:

1. $\Rightarrow$ 2.

Suppose $a \in X$ is an accumulation point of A, and G is an open set $a \in G$, containing only a finite number of points of A different from a. Then

$$B = \{a_1, a_2, \ldots, a_n\} = A \cap [G - \{a\}].$$

B is a finite subset of a T_1–space, hence, it is closed and $X - B$ is open. Let

$$H = (X - \text{B}) \cap G.$$

Then H is open, $a \in H$ and H do not contain any points of A different from a. Hence, a is not an accumulation point of A.

2. $\Rightarrow$ 1.

By definition of an accumulation point.

• PROBLEM 13-7

Prove, that if (X, T) and (Y, T') are homeomorphic and (X, T) is a T_1–space (or T_0–space), then so is (Y, T').

SOLUTION:

Let f denote a homeomorphism

$$f : X \rightarrow Y$$

and X be a T_1–space. A space (X, T) is T_1, if and only if every one-point subset of X is closed.

Let y represent any point of Y, $y \in Y$. The set $f^{-1}(y)$ is a one-point subset of X and since X is T_1, the set of $\{f^{-1}(y)\}$ is closed.

Since $f: X \to Y$ is a homeomorphism, it maps closed sets into closed sets. Therefore, for any $y \in Y$

$$\{y\} = \overline{\{y\}}.$$

Thus, (Y, T') is a T_1–space. Similarly, we can show that, if X is T_0, then so is Y.

● PROBLEM 13-8

Prove the following theorem:

THEOREM

Each T_2–space is a T_1–space.

SOLUTION:

DEFINITION OF T_2–SPACE (or a Hausdorff space or a separated space)

A topological space is a Hausdorff space if, for each pair of points $a \neq b$, two disjoint open sets A and B exist, such that

$$a \in A, \quad b \in B \quad A \cap B = \phi. \tag{1}$$

■

It is easy to see that each metric space is a T_2–space. Now, we shall show that each T_2–space is also a T_1–space. Suppose (X, T) is a T_2–space. Let $a \in X$ represent a given point. For each $x \in X, x \neq a$

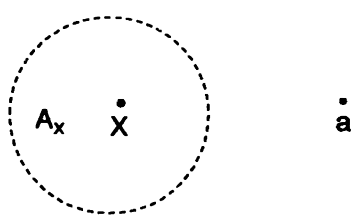

FIGURE 1

there is an open A_x, excluding the dotted circle in Figure 1, such that $x \in A_x$ and $a \notin A_x$. Thus

$$X - \{a\} = \bigcup_{x \neq a} A_x \tag{2}$$

and $X - \{a\}$ is an open set as a union of a family of open sets A_x.

Hence, $\{a\}$ is closed and X is a T_1–space.

• PROBLEM 13–9

Find a T_1–space which is not a T_2–space.

Show that the properties of being a T_1– and a T_2–space are hereditary.

SOLUTION:

Consider the set X, consisting of 0 and all points $^1/_n$, for $n = 1, 2, 3, \ldots$

$$X = \{0, 1, {}^1/_2, {}^1/_3, {}^1/_4, \ldots\} \tag{1}$$

We define the topology T on X: Sets containing 1 are open, if and only if they are complements of finite sets.

Sets which do not contain the point 1 are open, when they are open in the sense of the usual topology of real numbers.

Hence, each open set containing 0 is infinite.

Therefore, points 0 and 1 cannot be separated by two open disjoint sets. The space is T_1–space, but not T_2.

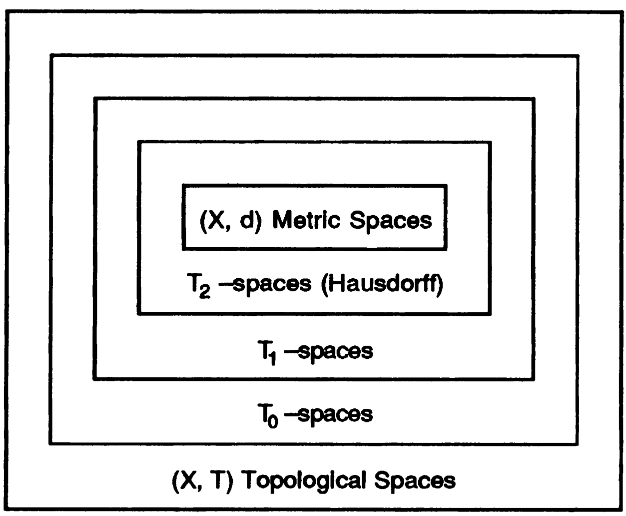

FIGURE 1

• PROBLEM 13–10

Show that the set X with the order topology is a T_2–space.

SOLUTION:

Let X represent any set totally ordered by $<$, $(X, <)$. Let S represent the family of subsets of X of the form

$$\{x : x < a\} \quad \text{or} \quad \{x : a < x\}$$

for all $a \in X$.

The family S forms a subbasis for a topology T on X, called the order topology induced by $<$. We shall show that (X, T) is a T_2–space.

Let $a, b \in X$ represent two distinct points. Since X is totally ordered, either $a < b$ or $b < a$. Suppose $a < b$. There are two possibilities now:

1. An element $c \in X$ exists, such that

$$a < c < b.$$

Then

$$\{x : x < c\} \quad \text{and} \quad \{x : c < x\}$$

are two disjoint neighborhoods of a and b respectively.

2. No $c \in X$ exists, such that

$$a < c < b.$$

Then

$$\{x : x < b\} \quad \text{and} \quad \{x : a < x\}$$

are disjoint neighborhoods of a and b respectively.

● PROBLEM 13–11

If

$$f : (X, T) \to (Y, T') \tag{1}$$

is onto and one-to-one, f^{-1} is continuous and X is a T_2–space, prove that Y is a T_2–space.

SOLUTION:

Let y_1 and y_2 represent any two distinct points of Y. Since f is one-to-one and onto, two distinct points of X exist, such that

$$x_1, x_2 \in X$$

$$f^{-1}(y_1) = x_1,\ \ f^{-1}(y_2) = x_2.$$

(X, T) is a Hausdorff space, therefore, there are two open sets $U_1, U_2 \subset X$, such that

$$x_1 \in U_1, \quad x_2 \in U_2, \quad U_1 \cap U_2 = \phi.$$

Since f is bijective,

$$f(U_1) \subset Y,\ \ f(U_2) \subset Y$$

$$f(U_1) \cap f(U_2) = \phi$$

Now, since f^{-1} is continuous, the function $(f^{-1})^{-1} = f$ maps open sets into open sets. Hence, $f(U_1), f(U_2) \in T'$ are open sets.

$$y_1 \in f(U_1),\ \ y_2 \in f(U_2)$$

We conclude that (Y, T') is a T_2–space.

If two spaces are homeomorphic and one of them is a T_2–space, then so is the other.

● PROBLEM 13–12

Prove the theorems:

1. $\begin{pmatrix} (X,T) \text{ a Hausdorff space} \\ A \text{ a finite subset of } X \end{pmatrix} \Rightarrow (A \text{ is closed}).$

2. $\begin{pmatrix} (X,T) \text{ a Hausdorff space} \\ A \subset X,\ x \text{ is a cluster point of } A \\ U \text{ a neighborhood of } X \end{pmatrix} \Rightarrow (U \cap A \text{ is infinite}).$

SOLUTION:

1. Let $x \in X$. We shall show that $\{x\}$ is closed. Indeed, let

$$y \in X - \{x\}$$

then a neighborhood U of y exists, such that

$$x \in U.$$

Hence,

$$U \subset X - \{x\}$$

and the set $X - \{x\}$ is open. Thus, $\{x\}$ is closed.

Any subset $\{x_1, ... x_n\}$ of a Hausdorff space is closed.

2. Suppose, on the contrary, that $U \cap A$ is finite. Then

$$U \cap [A - \{x\}]$$

is closed. Hence

$$U - [U \cap (A - \{x\}]$$

is open.

But

$$x \in U - [U \cap (A - \{x\})] = U - (A - \{x\}).$$

Then $U - (A - \{x\})$ is a neighborhood of x.

Since x is a cluster point of A, then

$$\{[U - (A - \{x\})] - \{x\}\} \cap A \neq \phi.$$

Contradiction!

• PROBLEM 13–13

Show, that if (X, T) is a Hausdorff space, then every convergent sequence in X has a unique limit.

SOLUTION:

Let (x_n) denote a convergent sequence with two limits a, b, such that $a \neq b$.

Since (X, T) is a Hausdorff space, the open sets U_1 and U_2 exist, such that

$$a \in U_1, \quad b \in U_2, \quad U_1 \cap U_2 = \phi.$$

But (x_n) converges to a. Thus,

$$\exists k_1 \quad \forall n > k_1 \quad x_n \in U_1.$$

Similarly,

$$\exists k_2 \quad \forall n > k_2 \quad x_n \in U_2.$$

But the sets U_1 and U_2 are disjoint. Contradiction. Hence, $a = b$.

● PROBLEM 13–14

Give an example of a regular space, which is not a T_1–space.

SOLUTION:

DEFINITION OF REGULAR SPACE

A topological space (X, T), is said to be regular if, given any closed subset $F \subset X$ and any point $x \in X$, such that $x \notin F$, there are open sets U and V, such that

$$F \subset U, \quad x \in V, \quad \text{and} \quad U \cap V = \phi \tag{1}$$

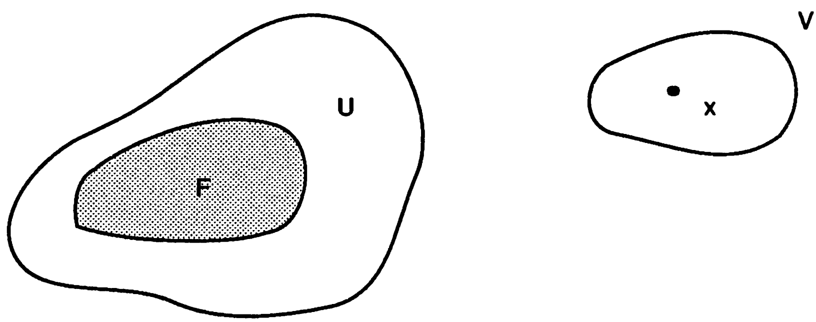

FIGURE 1

■

Consider the set $X = \{a, b, c\}$ with topology $T = \{\phi, X, \{a\}, \{b, c\}\}$. Note that the closed sets are $\phi, X, \{a\}, \{b, c\}$.

The topological space (X, T) is a regular space. But (X, T) is not a T_1–space. Set $\{c\}$ which is finite is not closed $\{c\} \neq \overline{\{c\}}$.

● PROBLEM 13–15

Let (Y, T_Y) denote the subspace of (X, T). Let $y \in Y$ and $A \subset Y$. Show, that if y does not belong to the T_Y–closure of A, then y does not belong to the T–closure of A.

SOLUTION:

By definition,

$$T_Y\text{–closure of } A = \overline{A} \cap Y \tag{1}$$

where $\overline{A}$ is the T–closure of A.

But

$$y \in Y. \tag{2}$$

Therefore, if

$$y \notin \overline{A} \cap Y \tag{3}$$

then

$$y \notin \overline{A}. \tag{4}$$

Note, that an equivalent definition of regular space is: a topological space (X, T) is called regular if, for every point x and every closed set F, such that $x \notin F$, an open set G exists, such that

$$x \in G \quad \text{and} \quad \overline{G} \cap F = \phi.$$

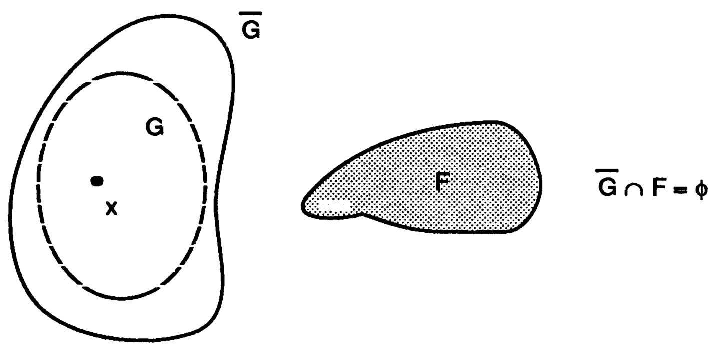

FIGURE 1

● PROBLEM 13–16

Show that every subset of a regular space is regular, i.e., the property of being a regular space is hereditary.

SOLUTION:

Let (X, T) denote a regular space and (A, T_A) its subspace. See Figure 1. Let $y \in A$ and let F represent a T_Y–closed subset of A, such that $y \notin F$. Hence, by Problem 13–15,

$$y \notin \overline{F}$$

where $\overline{F}$ is the T–closure of F. Space (X, T) is regular. Two open sets G and H exist, such that

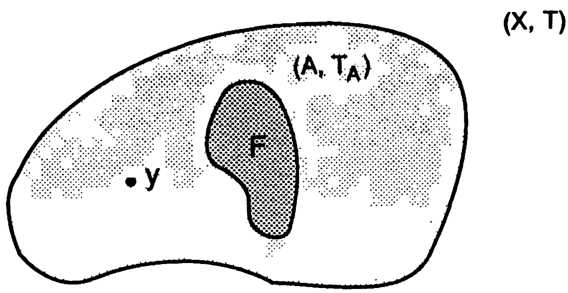

FIGURE 1

$$\overline{F} \subset G, \quad y \in H, G \cap H = \phi .$$

But

$$G \cap A \text{ and } H \cap A \text{ are } T_A\text{–open}$$

subsets of A.

$y \in A \cap H$ because $y \in A$ and $y \in H$. Sets $A \cap G$ and $A \cap H$ are disjoint because $G \cap H = \phi$.

Also, since

$$F \subset A \text{ and } F \subset \overline{F} \subset G \Rightarrow F \subset A \cap G .$$

Thus, (A, T_A) is also regular.

• PROBLEM 13-17

Show that a space (X, T) is regular, if and only if given any $x \in X$ and any neighborhood U of x, $x \in U \in T$, there is a neighborhood V of x, such that

$$\overline{V} \subset U .$$

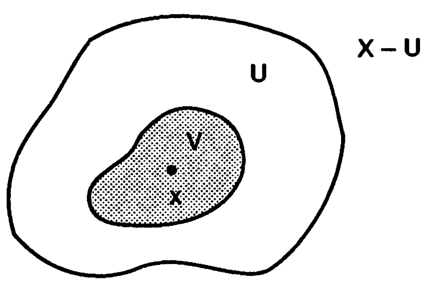

FIGURE 1

SOLUTION:

Suppose (X, T) is regular. Let U represent the neighborhood of x. Set $X - U$

is closed and $x \notin X - U$. Hence, open sets V and Z exist, such that

$$x \in V, \quad X - U \subset Z, \quad V \cap Z = \phi.$$

Since $X - U \subset Z$, we have $X - Z \subset U$. Also, since $V \cap Z = \phi$, we have

$$V \subset X - Z \subset U.$$

Hence,

$$V \subset \overline{V} \subset X - Z \subset U.$$

Now, suppose $x \in X$ and U are any neighborhood of x. Then, a neighborhood V of x exists, such that

$$\overline{V} \subset U.$$

Let $x \in X$ and F represent any closed subset of X, such that $x \notin F$.

$X - F$ is a neighborhood of x. Then V exists, such that, V is open, $x \in V$

$$\overline{V} \subset X - F.$$

Set $X - \overline{V}$ is open and $F \subset X - \overline{V}$, also V is an open subset which contains x. Since $V \subset \overline{V}$

$$V \cap (X - \overline{V}) = \phi.$$

Thus, V and $X - \overline{V}$ are the sets that we are looking for.

(X, T) is regular.

• PROBLEM 13–18

Show that every T_3–space is also a T_2–space.

SOLUTION:

DEFINITION OF A T_3–SPACE

A regular T_1–space is called a T_3–space.

■

Let (X, T) denote a T_3–space. We shall show that (X, T) is also a Hausdorff space.

Let $a, b \in X$ represent distinct points. Space (X, T) is a T_1–space, therefore, $\{a\}$ is a closed set. Since a and b are distinct,

$$b \in \{a\}$$

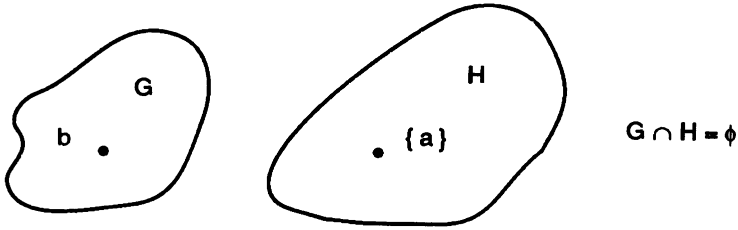

FIGURE 1

Since (X, T) is a regular space, two disjoint open sets G and H exist, such that

$$b \in G \quad \text{and} \quad \{a\} \subset H$$

Hence, a and b belong to two disjoint open sets G and H. (X, T) is a Hausdorff space (T_2–space).

● PROBLEM 13–19

Show that any metric space (X, d) is a T_3–space.

SOLUTION:

Let (X, d) denote a metric space and U represent any neighborhood of $x \in X$. There is a ball $B(x, r)$, such that

$$x \in B(x, r) \subset U. \tag{1}$$

Let us take any number r'

$$0 < r' < r. \tag{2}$$

Then

$$B(x, r') \subset B(x, r) \tag{3}$$

and

$$\overline{B(x, r')} = \{y : d(x, y) \leq r'\} \subset B(x, r) \subset U. \tag{4}$$

Hence, for any $x \in X$ and any neighborhood U of x, there is a neighborhood $B(x, r')$ of x, such that

$$\overline{B(x, r')} \subset U.$$

According to Problem 13–17, we conclude that (X, d) is regular. Since (X, d) is also T_1, any metric space is a T_3–space.

• PROBLEM 13-20

We shall define a topology on the set of real numbers R by giving an open neighborhood system. Let $x \neq 0$, then P_x is the family of all open intervals which contain x. For $x = 0$, we define P_0 as the family of all sets of the form $]-a, a[- \{^1/_n\}$, where n is a positive integer. Show that (R, T) defined above is T_2, but not T_3.

SOLUTION:

First, we show that (R, T) is T_2. Let $x \in R$ and $y \in R$, $x \neq y$. If both x and y are different from 0, then

$$U =]x - \frac{|x-y|}{2}, x + \frac{|x-y|}{2}[$$

$$V =]y - \frac{|x-y|}{2}, y + \frac{|x-y|}{2}[$$

Bot U and V are open sets and

$$x \in U, \quad y \in V, \quad U \cap V = \phi.$$

If one point is 0, i.e., $x = 0$, then

$$U =]- \frac{|y|}{2}, \frac{|y|}{2}[- \{\frac{1}{n}\}.$$

Hence, (R, T) is a Hausdorff space (i.e., a T_2–space).

Now, we shall show that (R, T) is not T_3. Let $x = 0$ ad $F = \{^1/_n\}$, where n is a positive integer. The set F is closed.

Let V represent any neighborhood of $x = 0$

$$V =]-a, a[- \{^1/_n\}.$$

No open set U exists, such that $F \subset U$ and $V \cap U = \phi$.

Hence, (R, T) is not regular and, therefore, not T_3.

• PROBLEM 13-21

Show that the property of being T_3 is hereditary, that is, that any subspace of a T_3–space is a T_3–space.

SOLUTION:

Let Y denote a subspace of a T_3–space (X, T). Let F be closed in Y and

$x \in Y$ and $x \notin F$.

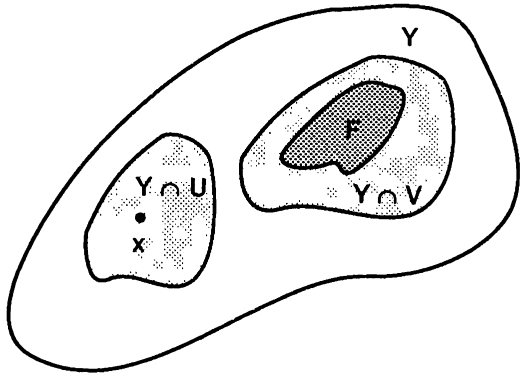

FIGURE 1

Since F is closed in Y,

$$F = Y \cap F' \tag{1}$$

where F' is a closed subset of X. Then $x \notin F'$. Space X is regular, hence, the open sets U and V in X exist, such that

$$x \in U, \quad F' \subset V, \quad U \cap V = \phi. \tag{2}$$

The sets $Y \cap U$ and $Y \cap V$ are open in Y and

$$x \in Y \cap U, \quad F \subset Y \cap V, \quad (Y \cap U) \cap (Y \cap V) = \phi. \tag{3}$$

Hence, space (Y, T_Y) is regular. Since (Y, T_Y) is a subspace of a T_1–space (X, T), it is also T_1. Therefore, (Y, T_Y) is a regular and a T_1–space, i.e., a T_3–space.

● PROBLEM 13–22

Show that any space (X, T), containing more than one point with the indiscrete topology, is normal.

SOLUTION:

DEFINITION OF NORMAL SPACE

A topological space (X, T) is said to be normal if, given any two disjoint closed sets F_1 and F_2 in X, there are disjoint open sets U and V, such that

$$F_1 \subset U \quad \text{and} \quad F_2 \subset V$$

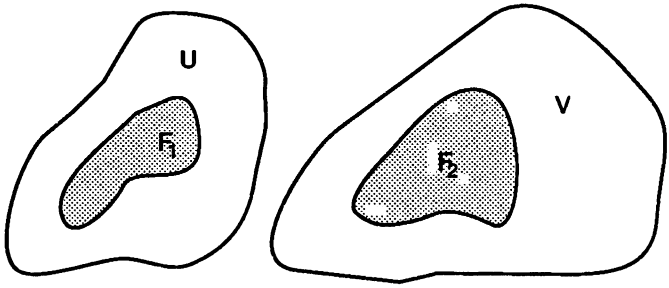

FIGURE 1

■

The indiscrete topology consists of two sets X and ϕ.

$$T = \{X, \phi\}.$$

Hence, the only closed sets are X and ϕ because $X - \phi = X$ and $X - X = \phi$.

Thus, there are no non-empty disjoint closed subsets of X. The space is normal.

It is easy to show that the space with discrete topology is normal. In this topology, each set is closed and open.

● PROBLEM 13-23

Prove this theorem:

THEOREM

A topological space (X, T) is normal iff, for every closed set F and every open set H containing F, an open set U exists, such that

$$F \subset U \subset \overline{U} \subset H. \tag{1}$$

■

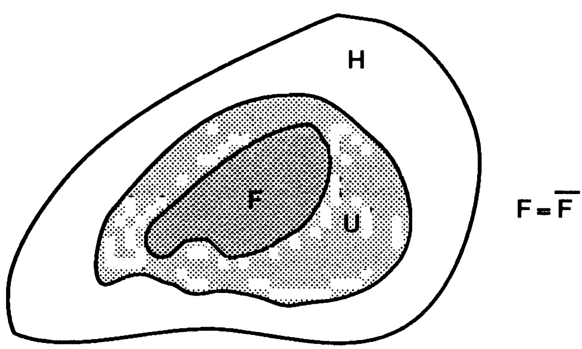

FIGURE 1

SOLUTION:

$\Rightarrow$ Let (X, T) denote a normal space. Let F represent a closed set and H an open set, such that

$$F \subset H \tag{2}$$

The set $X - H$ is closed and

$$F \cap (X - H) = \phi. \tag{3}$$

F and $X - H$ are two disjoint closed sets. Hence, the open sets U and U' exist, such that

$$F \subset U, \quad X - H \subset U', \quad U \cap U' = \phi. \tag{4}$$

Since

$$U \cap U' = \phi, \text{ we have } U \subset X - U' \tag{5}$$

also, since

$$X - H \subset U', \quad \text{we have } X - U' \subset H. \tag{6}$$

Set $X - U'$ is closed, hence, we conclude

$$F \subset U \subset \overline{U} \subset X - U' \subset H. \tag{7}$$

$\Leftarrow$ Let F_1 and F_2 denote disjoint closed sets. Then

$$F_1 \subset X - F_2 \tag{8}$$

and $X - F_2$ is open. An open set exists, such that

$$F_1 \subset U \subset \overline{U} \subset X - F_2. \tag{9}$$

But, since $\overline{U} \subset X - F_2$, we must have $F_2 \subset X - \overline{U}$. Also since $U \subset \overline{U}$, we have $U \cap (X - \overline{U}) = \phi$.

Thus, since $X - \overline{U}$ is open $F_1 \subset U$, $F_2 \subset X - \overline{U}$ and $U \cap (X - \overline{U}) = \phi$ where U and $X - \overline{U}$ are open sets.

● PROBLEM 13–24

Show that every metric space is normal.

SOLUTION:

Let (X, d) denote a metric space. Metric d induces the topology T on X. To show that (X, T) is normal, we shall apply the separation axiom.

THEOREM (SEPARATION AXIOM)

Let A_1 and A_2 represent closed disjoint subsets of a metric space (X, d). Then the disjoint open sets U_1 and U_2 exist, such that

$$A_1 \subset U_1 \quad \text{and} \quad A_2 \subset U_2$$

■

From this, we conclude that every metric space is normal.

● PROBLEM 13-25

Consider the set $X = \{a, b, c\}$ with the topology

$$T = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}. \tag{1}$$

Is (X, T) a T_1–space, a regular space, or a normal space?

SOLUTION:

The closed sets are

$$X, \phi, \{b, c\}, \{a, c\}, \{c\} \tag{2}$$

Not every singleton subset of X is closed. For example,

$$\{a\} \neq \overline{\{a\}}. \tag{3}$$

Hence, the space (X, T) is not T_1. Also, (X, T) is not a regular space. Take a closed subset $\{c\}$ and $a \notin \{c\}$. Then, the only open set, which contains $\{c\}$ is the whole space X, which contains a. We shall show that (X, T) is a normal space.

Let F_1 and F_2 represent disjoint closed subsets of X. Then one of them, say F_1 is the empty set ϕ.

The sets ϕ and X are disjoint open sets and

$$\phi = F_1 \subset \phi; \quad F_2 \subset X.$$

Thus, (X, T) is a normal space.

● PROBLEM 13-26

Show that every T_4–space is also a T_3–space. See Figure 1.

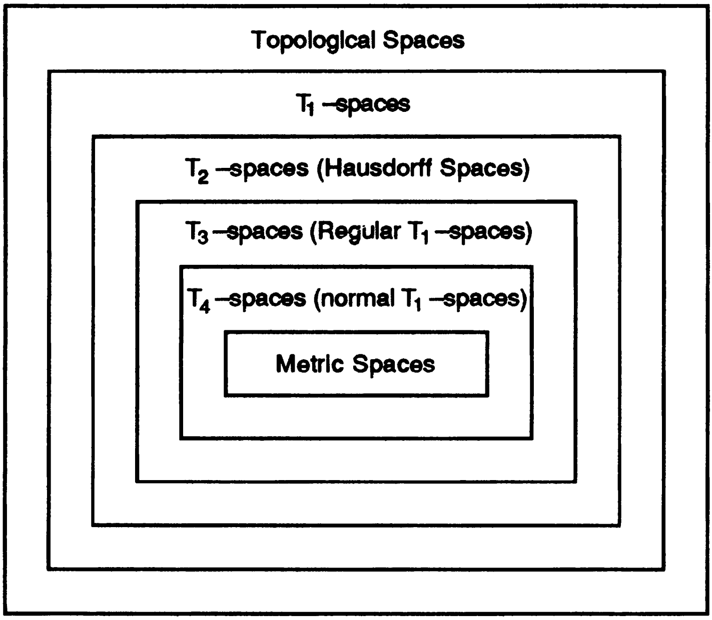

FIGURE 1

SOLUTION:

DEFINITION OF A T_4–SPACE

A normal space which is also a T_1–space is called a T_4–space.

■

Let (X, T) denote a T_4–space. Hence, (X, T) is normal and T_1. Suppose F is a closed subset of X and $a \notin F$. Since (X, T) is T_1, the singleton set $\{a\}$ is closed. Sets F and $\{a\}$ are closed and disjoint. Since (X, T) is normal, the open sets U_1 and U_2 exist, such that

$$\{a\} \subset U_1, \quad F \subset U_2, \quad U_1 \cap U_2 = \phi.$$

Therefore (X, T) is regular and T_3.

The diagram illustrates the relationship between different kinds of topological spaces, defined in this chapter.

● **PROBLEM 13-27**

Prove that if Y is a closed subset of a T_4–space (X, T), then the subspace (Y, T_Y) is also T_4–space.

SOLUTION:

Since every subspace of a T_1–space is T_1 and (X, T) is T_1 also, Y is a T_1–space. Since Y is closed, a subset F of Y is closed in Y, if and only if F is closed in X. Hence, if F_1 and F_2 are disjoint closed subsets of Y, they are also disjoint closed subsets of X.

Thus, the open sets U_1 and U_2 exist, such that

$$F_1 \subset U_1, \quad F_2 \subset U_2 \text{ and } U_1 \cap U_2 = \phi.$$

Then

$$F_1 \subset U_1 \cap Y, \quad F_2 \subset U_2 \cap Y,$$

and $U_1 \cap Y$ and $U_2 \cap Y$ are disjoint subsets of Y, open in Y. Since (Y, T_Y) is T_1 and normal, it is T_4.

● PROBLEM 13–28

Suppose that the space (X, T) is homeomorphic to the space (Y, T'), and that X is T_4. Prove that Y is T_4.

SOLUTION:

We have already shown that if a space X is homeomorphic to a space Y and X is T_1, then Y is T_1.

Now, suppose X is a normal space and $f: X \to Y$ is a homeomorphism.

Let G_1 and G_2 represent any two disjoint closed subsets of Y. Then $f^{-1}(G_1)$

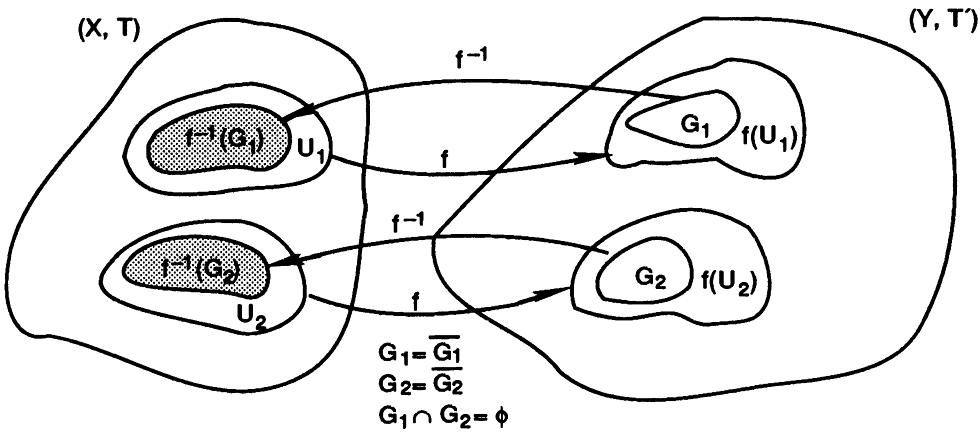

FIGURE 1

and $f^{-1}(G_2)$ are disjoint closed subsets of X. Since X is normal, there are two disjoint open sets U_1 and U_2 in X, such that

$$f^{-1}(G_1) \subset U_1 \quad f^{-1}(G_2) \subset U_2$$

The sets $f(U_1)$ and $f(U_2)$ are open in Y and

$$G_1 \subset f(U_1), \quad G_2 \subset f(U_2)$$

$$f(U_1) \cap f(U_2) = \phi.$$

Hence, (Y, T') is normal and T_4.

● PROBLEM 13-29

Describe the notion of the extension of a continuous function defined on a topological subspace.

SOLUTION:

The problem of extension of a function is one of the central and most difficult questions in all of topology. Suppose (X, T) is a topological space and (Y, T_Y) is its topological subspace.

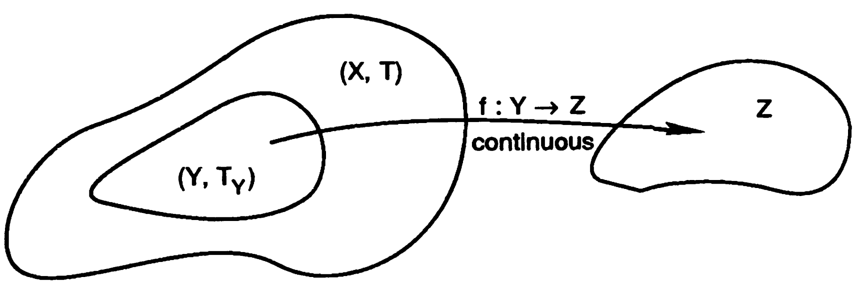

FIGURE 1

Let $f: Y \to Z$ denote a continuous function from Y into some space (Z, T'). The question is: does a continuous function $F: X \to Z$ exist, such that

$$F(y) = f(y) \text{ for each } y \in Y,$$

that is, $F \mid Y = f$?

In most cases the answer to this question is unknown. Only normal spaces have certain extension properties.

Here is one of the most important theorems in topology:

URYSOHN'S LEMMA

If a topological space (X, T) is normal, then, given any disjoint closed non-empty subsets A and B of X, there is a continuous function $f: X \to Z$, $Z = [0, 1]$, (Z has the absolute value topology), such that

$$\text{for every } a \in A, f(a) = 0$$

and

$$\text{for every } b \in B, f(b) = 1.$$

■

Prove that the converse of Urysohn's lemma is true.

SOLUTION:

Suppose (X, T) has the property described in Urysohn's lemma. Let A and B represent closed disjoint non-empty subsets of X. There is a continuous function $f: X \to [0, 1]$, such that for all $a \in A$, $f(a) = 0$ and for all $b \in B$, $f(b) = 1$.

Sets

$$U = \{x : 0 \le x < {}^1/_2\}$$

$$U' = \{x : {}^1/_2 < x \le 1\}$$

are disjoint open subsets of $[0, 1]$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(U')$ are disjoint open subsets of X, such that

$$A \subset f^{-1}(U)$$

$$B \subset f^{-1}(U').$$

Hence, (X, T) is normal.

● PROBLEM 13–31

Let (X, T) denote a T_4–space which contains more than one point. Show that a non-constant continuous function $f: X \to [0, 1]$ exists.

SOLUTION:

Space X contains at least two points, for example, x and y. Since (X, T) is a T_4–space, it must be normal and T_1. For any T_1–space,

$$\{x\} = \{\overline{x}\}$$

$$\{y\} = \{\overline{y}\}$$

That is, any one-point subset is closed. The sets $\{x\}$ and $\{y\}$ are disjoint closed non-empty subsets of X.

Therefore, since X is normal, a continuous function

$$f : X \to [0, 1]$$

exists, such that

$$f(x) = 0 \quad \text{and} \quad f(y) = 1.$$

Function f is a continuous non-constant function from X into $[0, 1]$.

• PROBLEM 13–32

Prove this generalization of Urysohn's lemma:

Let A and B represent disjoint closed non-empty subsets of a normal space (X, T). Then a continuous function

$$f : X \to [a, b\}$$

exists, such that

$$f(x) = a \quad \text{for all} \quad x \in A$$

$$f(x) = b \quad \text{for all} \quad x \in B.$$

SOLUTION:

Since (X, T) is a normal space, we can apply Urysohn's lemma (Problem 13–30).

Function f^*, which is continuous exists, such that

$$f^* : X \to [0, 1]$$

and for all $x \in A$, $f(x) = 0$ and for all $x \in B$, $f(x) = 1$.

Consider function f, defined by

$$f(x) = a + (b - a) f^*(x).$$

Since f^* is continuous, so is f. Furthermore,

$$f(x) = \begin{cases} a & \text{for all } x \in A \\ b & \text{for all } x \in B \end{cases}.$$

● PROBLEM 13–33

Show that Urysohn's lemma is a special case of Tietze's extension theorem.

TIETZE'S EXTENSION THEOREM

Let (X, T) denote a T_4–space and Y any closed subset of X. If f is any continuous function $f: Y \to R$ (where R is the space of real numbers with the absolute value topology), then there is a continuous extension F of f, from X into R

$$F : X \to R.$$

■

SOLUTION:

Let (X, T) denote a T_4–space and let A and B represent disjoint closed non-empty subsets of X. We define

$$Y = A \cup B \quad \text{and} \quad f : Y \to R$$

by $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

Function f is continuous. Hence, according to Tietze's theorem, it can be extended.

● PROBLEM 13–34

Let (X, T) denote any T_4–space and A represent any closed subset of X. Show that for any continuous function

$$f : A \to R^n$$

a continuous extension F of f exists, such that

$$F : X \to R^n$$

Note that R^n is an n–dimensional Cartesian product of R.

See Figure 1.

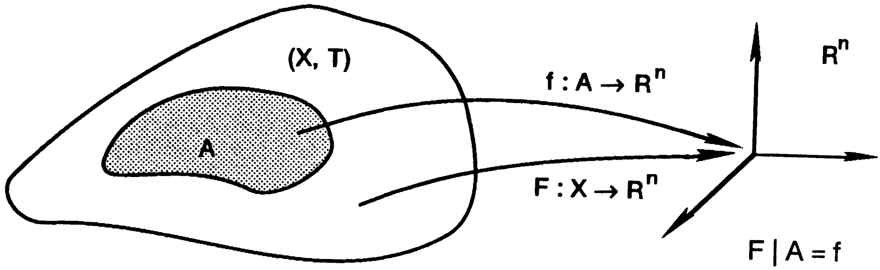

FIGURE 1

SOLUTION:

f is a continuous function of A into R^n. Thus, it can be written as

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x)).$$

Each of the functions $f_1, \ldots, f_n$ is continuous and maps A into R. According to Tietze's extension theorem, for each of the functions f_i, $i = 1, \ldots, n$, there exists a continuous extension F_i of f_i, $i = 1, 2, \ldots, n$, from X into R. We define

$$F(x) = (F_1(x), \ldots, F_n(x)).$$

Since all functions $F_1, F_2, \ldots, F_n$ are continuous, so is F. Function F is a continuous extension of f from X into R^n.

○ PROBLEM 13-35

Let R denote the space of real numbers with the absolute value topology and Z the set of integers, $Z \subset R$, with subspace topology. Let R^2 represent the plane with product topology and $Z^2 \subset R^2$ the set of all pairs (m, n), where m and n are integers, with subspace topology.

Using this "scenario," prove the existence of a continuous function

$$F : R \rightarrow R^2.$$

SOLUTION:

Consider any function

$$f : Z \rightarrow Z^2.$$

In both cases, the subspace topology is the discrete topology. Hence, any function from Z into Z^2 is continuous. Sets Z and Z^2 have the same cardinal number. It is possible to find function $f : Z \rightarrow Z^2$, which is one-to-one and

continuous.

According to Problem 13–34, f has a continuous extension F from R into R^2.

● PROBLEM 13–36

Consider the class of real-valued functions

$$F = \{f_1(x) = \sin x, f_2(x) = \sin 2x, \ldots, f_n(x) = \sin nx, \ldots\}$$

defined in R. Show that F does not separate points of R.

SOLUTION:

DEFINITION

A class $F = \{f_\omega : \omega \in \Omega\}$ of functions from X into Y is said to separate points if, for any pair of distinct points $a, b \in X$, a function $f \in F$ exists, such that $f(a) \neq f(b)$.

■

Consider a pair of distinct points 0 and π.

$$\sin 0 = \sin 2 \cdot 0 = \sin 3 \cdot 0 = \ldots \qquad = 0$$

$$\sin \pi = \sin 2\pi = \sin 3\pi = \ldots \qquad = 0$$

Hence class F does not separate points.

● PROBLEM 13–37

Let $C(X, R)$ denote the class of all real-valued continuous functions, defined on a topological space (X, T). Show that, if the class $C(X, R)$ separates points, then (X, T) is a Hausdorff space.

SOLUTION:

Let $a, b \in X$ represent distinct points. Since $C(X, R)$ separates points, a continuous function

$$f : X \to R$$

exists, such that $f(a) \neq f(b)$.

There are two open disjoint subsets U_1 and U_2 of R, such that

$$f(a) \in U_1 \text{ and } f(b) \in U_2.$$

Since f is a continuous function, sets $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open and disjoint.

$$a \in f^{-1}(U_1)$$

$$b \in f^{-1}(U_2).$$

Hence, (X, T) is a Hausdorff space (T_2–space).

● PROBLEM 13-38

Prove that a completely regular space is regular.

SOLUTION:

DEFINITION OF COMPLETELY REGULAR SPACE

A topological space (X, T) is completely regular if, for any closed subset F of X and any $a \in X$, such that $a \notin F$, a continuous function $f : X \to [0, 1]$ exists, such that for every

$$x \in F, \quad f(x) = 1 \quad \text{and} \quad f(a) = 0.$$

■

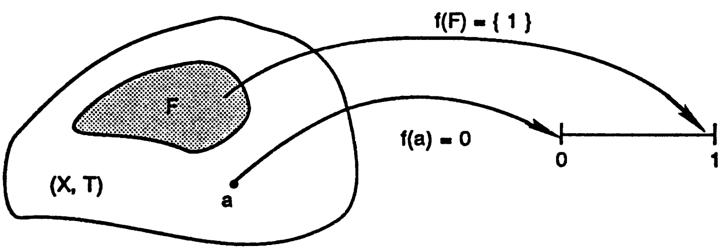

FIGURE 1

Now we show that completely regular space is also regular.

Let F represent a closed subset of X and $a \in X$ a point which does not belong to F.

By hypothesis, a continuous function

$$f : X \to [0, 1]$$

exists, such that $f(F) = \{1\}$ and $f(a) = 0$. An interval $[0, 1]$ is a Hausdorff space. Hence, two open disjoint subsets U_1 and U_2 of $[0, 1]$ exist, such that

$$0 \in U_1 \quad \text{and} \quad 1 \in U_2.$$

Since f is continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open. These subsets are disjoint and such that

$$a \in f^{-1}(U_1), \quad F \subset f^{-1}(U_2).$$

Hence,·(X, T) is regular.

● PROBLEM 13–39

> Show that the property of being a completely regular space is hereditary, i.e., every subspace of a completely regular space is completely regular.

SOLUTION:

Let (X, T) denote a completely regular space and let (Y, T_Y) be its subspace.

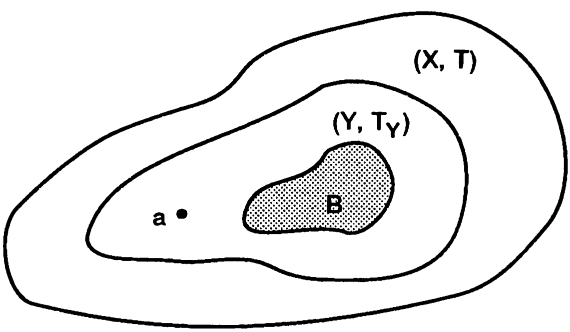

FIGURE 1

Let B be closed in Y and a represent a point in Y, such that $a \notin B$. Then

$$B = Y \cap D$$

where D is a closed set in X. Since $a \in Y$ and $a \notin B$, we have

$$a \notin D.$$

Since (X, T) is completely regular, a continuous function $f: X \to [0, 1\}$ exists, such that

$$f(a) = 0 \quad \text{and for each } x \in D \quad f(x) = 1.$$

Then $f \mid Y$ is a continuous function.

$$f \mid Y: Y \to [0, 1]$$

such that $f \mid Y(a) = 0$ and for each $x \in B, f \mid Y(x) = 1$.

Thus, (Y, T_Y) is a completely regular space.

• PROBLEM 13-40

Consider the drawing of Problem 13–26. Where would you locate on this drawing the Tychonoff spaces?

SOLUTION:

DEFINITION OF TYCHONOFF SPACE

A completely regular T_1–space is called a Tychonoff space.

■

In Problem 13–38, we proved that a completely regular space is also a regular space.

Hence, only a Tychonoff space is a T_3–space.

Any normal T_1–space (i.e., T_4–space) is a Tychonoff space by virtue of Urysohn's lemma.

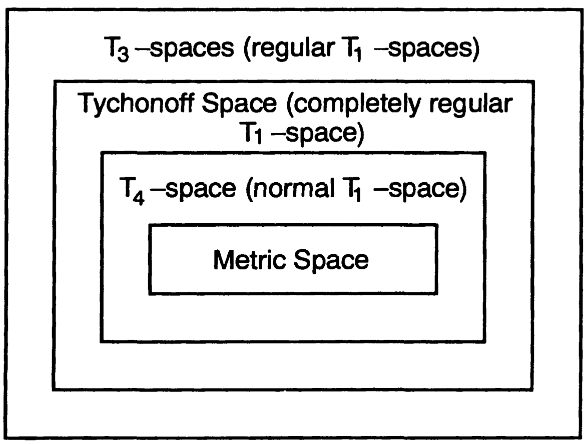

FIGURE 1

• PROBLEM 13-41

Prove the following:

THEOREM

The class $C(X, R)$ of all real-valued continuous functions on a completely regular T_1–space X separates points.

■

SOLUTION:

Let a and b represent any distinct points in X. Since X is T_1, the set $\{a\}$ is closed. Points a and b are distinct, hence,

$$b \notin \{a\}.$$

Space (X, T) is completely regular. Hence, a real-valued continuous function f on X exists, such that

$$f(b) = 0 \quad \text{and} \quad f(a) = 1.$$

This function separates points a and b

$$f(a) \neq f(b).$$

CHAPTER 14

CARTESIAN PRODUCTS

Cartesian Product, Projections	14–1, 14–2, 14–3, 14–4, 14–5, 14–6
Invariants of Cartesian Product	14–7, 14–8
Diagonal	14–9, 14–10, 14–11
Generalized Cartesian Product	14–12, 14–13
Continuity of Maps	14–14, 14–15
Cartesian Product of Metric Spaces	14–16, 14–17

• PROBLEM 14–1

Let (X_1, T_1) and (X_2, T_2) denote topological spaces.

DEFINITION OF OPEN SETS IN $X_1 \times X_2$

A set $A \subset X_1 \times X_2$ is called open in $X_1 \times X_2$ iff it is the union of Cartesian products $G \times H$, where $G \in T_1$, and $H \in T_2$

■

Prove the following:

THEOREM

The Cartesian product of two topological spaces is a topological space.

■

SOLUTION:

From the definition, we conclude that the family of all sets $G \times H$ is a base of $X_1 \times X_2$, where $G \in T_1$ and $H \in T_2$.

From the identity

$$(G_1 \times H_1) \cap (G_2 \times H_2) = (G_1 \cap G_2) \times (H_1 \cap H_2) \tag{1}$$

we see that, if $G_1, G_2 \in T_1$ and $H_1, H_2 \in T_2$, then the intersection $(G_1 \times H_1) \cap (G_2 \times H_2)$ of open sets $G_1 \times H_1$ and $G_2 \times H_2$ is an open set.

Since

$$\left(\bigcup_{\alpha} G_\alpha\right) \times \left(\bigcup_{\beta} H_\beta\right) = \bigcup_{\alpha,\beta} (G_\alpha \times H_\beta) \tag{2}$$

we conclude that the union of an arbitrary family of open sets in $X_1 \times X_2$ is open.

Thus, the Cartesian product of two topological spaces is a topological space.

• PROBLEM 14–2

Show that in R^2 the usual topology agrees with the definition of Problem 14–1.

SOLUTION:

Let A denote any open set in R^2. It can be easily shown that the collection of open squares with the sides parallel to the x and y axes forms a basis for

R^2 space.

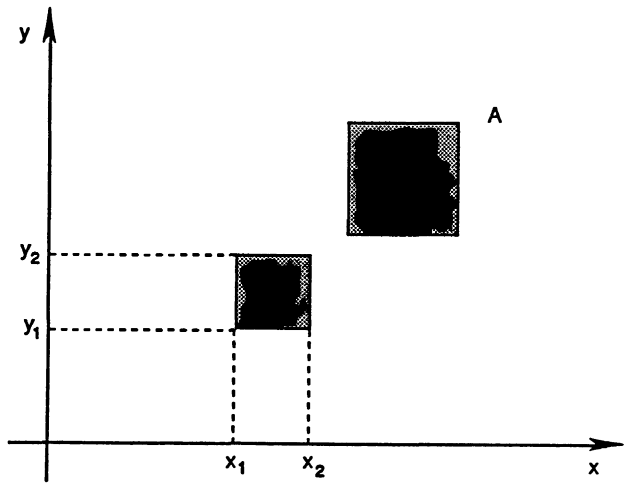

FIGURE 1

Thus

$$A = \bigcup_{\alpha} S_{\alpha}, \tag{1}$$

where each S_α is an open square with the sides parallel to the x and y axes. Since the squares' boundaries are open as indicated by the dashed lines, the boundaries are not included in S_α.

Observe that each square can be represented as the Cartesian product of two open intervals

$$S_\alpha =]\, x_1, x_2 \,[\times]\, y_1, y_2 \,[\tag{2}$$

Hence, the collection of sets of the type $]\, x_1, x_2 \,[\times]\, y_1, y_2 \,[$ forms a basis of R^2.

● PROBLEM 14–3

Prove that

THEOREM

If $\{A_\alpha\}$ is a base of X_1, and $\{B_\beta\}$ is a base of X_2, then $\{A_\alpha \times B_\beta\}$ is a base of $X_1 \times X_2$.

■

SOLUTION:

Suppose $W \subset X_1 \times X_2$ is an open set in $X_1 \times X_2$. Then, by Problem 14–1,

W is the union of Cartesian products $E_\gamma \times F_\delta$ where E_γ and F_δ are open subsets of X_1 and X_2 respectively. Thus

$$W = \cup(E_\gamma \times F_\delta) \tag{1}$$

Since $\{A_\alpha\}$ is a base of X_1 and $\{B_\beta\}$ is a base of X_2, we have for each E_γ

$$E_\gamma = \cup A_{\alpha,\gamma} \tag{2}$$

and for each F_δ

$$F_\delta = \cup B_{\beta,\delta} \tag{3}$$

where $A_{\alpha,\gamma} \in \{A_\alpha\}$, $B_{\beta,\delta} \in \{B_\beta\}$. Hence

$$E_\gamma \times F_\delta = (\cup A_{\alpha,\gamma}) \times (\cup B_{\beta,\delta}) = \cup (A_{\alpha,\gamma} \times B_{\beta,\delta}). \tag{4}$$

Therefore

$$W = \cup(E_\gamma \times F_\delta) = \cup (A_\alpha \times B_\beta). \tag{5}$$

The set of $\{A_\alpha \times B_\beta\}$ is the base of $X_1 \times X_2$.

● PROBLEM 14-4

Show that

THEOREM

The projections

$$\pi_1 : X \times Y \to X; \quad \pi_1(x, y) = x \tag{1}$$

$$\pi_2 : X \times Y \to Y; \quad \pi_2(x, y) = y \tag{2}$$

are continuous mappings.

■

SOLUTION:

Let A be an open subset of X. Then

$$\pi_1^{-1}(A) = A \times Y. \tag{3}$$

Since both A and Y are open sets, $\pi_1^{-1}(A)$ is an open set and π_1 is a continuous mapping. Similarly we show that π_2 is a continuous mapping, since for any open subset B of Y

$$\pi_2^{-1}(B) = X \times B \tag{4}$$

$X \times B$ is an open set.

This result holds for the generalized Cartesian product of any family of topological spaces.

● PROBLEM 14–5

Let

$$f : T \to X \times Y \tag{1}$$

where

$$f(t) = (f_1(t), f_2(t)). \tag{2}$$

Show that f is continuous iff f_1 and f_2 are continuous. Also, f is continuous at t_0 iff f_1 and f_2 are continuous at t_0.

SOLUTION:

Suppose that f is continuous at t_0. Since

$$f_1(t) = \pi_1[f(t)] \tag{3}$$

by Problem 14–4, we conclude that f_1 is continuous at t_0.

Now, suppose that f_1 and f_2 are continuous at t_0. Let $A \subset X \times Y$ be open and let

$$t_0 \in f^{-1}(A). \tag{4}$$

We shall show that

$$t_0 \in Int\, f^{-1}(A). \tag{5}$$

We can apply the following property:

If S is a subbase of Y and $f : X \to Y$ and if $f^{-1}(D)$ is open for each $D \in S$, then f is continuous.

Thus, we can assume that A belongs to a subbase of $X \times Y$. Setting

$$A = G \times Y \tag{6}$$

we obtain

$$f^{-1}(A) = f_1^{-1}(G). \tag{7}$$

Therefore

$$t_0 \in f_1^{-1}(G). \tag{8}$$

But f_1 is continuous at t_0, thus

$$t_0 \in Int\, f_1^{-1}(G) = Int\, f^{-1}(A). \tag{9}$$

● PROBLEM 14-6

Let

$$f : X \to Y$$

$$g : W \to Z \tag{1}$$

where X, Y, W and Z are topological spaces.

Show that the product mapping

$$h = f \times g : X \times W \to Y \times Z \tag{2}$$

is continuous iff f and g are continuous.

SOLUTION:

We have

$$h(x, w) = (f(x), g(w)). \tag{3}$$

Let w_0 represent a given element of W. Then

$$h_1(x) =: h(x, w_0) = (f(x), g(w_0)) \tag{4}$$

and we obtain

$$f = \pi_1 \circ h_1. \tag{5}$$

We shall apply the following.

THEOREM

A continuous mapping of two variables is continuous with respect to each variable.

■

Hence, if h is continuous, then so is f. By the same argument g is continuous.

Now, suppose that f and g are continuous. Then, by applying

$$h^{-1}(A \times B) = f^{-1}(A) \times g^{-1}(B) \tag{6}$$

we conclude that h is continuous.

● PROBLEM 14-7

We shall investigate the properties which are invariants of Cartesian multiplication.

Prove:

THEOREM

The product of two closed sets is a closed set. ■

SOLUTION:

Let $A \subset X$ be closed in X and $B \subset Y$ be closed in Y. We have

$$(X \times Y) - (A \times B) = [(X - A) \times Y] \cup [X \times (Y - B)]. \tag{1}$$

Thus, the set $(X \times Y) - (A \times B)$ is the union of two open sets and is therefore an open set.

Similarly it can be shown that if both spaces are T_2–spaces, then their Cartesian product is a T_2–space.

Also regularity is invariant under Cartesian multiplication. But normality is not invariant under Cartesian multiplication.

● PROBLEM 14-8

Show that the Cartesian product of two T_2–spaces is a T_2–space.

SOLUTION:

Suppose that X and Y are T_2–spaces. Let $z_1 \in X \times Y$ and $z_2 \in X \times Y$, such that

$$z_1 \neq z_2 \tag{1}$$

$$z_1 = (x_1, y_1) \qquad z_2 = (x_2, y_2) \tag{2}$$

Since $z_1 \neq z_2$, we must have either

$$x_1 \neq x_2 \tag{3}$$

or

$$y_1 \neq y_2 \tag{4}$$

Suppose that $x_1 \neq x_2$. Since X is a T_2–space, two open sets A_1 and A_2 exist, such that,

$$A_1 \subset X, \; A_2 \subset X \tag{5}$$

$$x_1 \in A_1, x_2 \in A_2 \tag{6}$$

$$A_1 \cap A_2 = \phi. \tag{7}$$

Therefore

$$z_1 \in A_1 \times Y \quad z_2 \in A_2 \times Y. \tag{8}$$

The sets $A_1 \times Y$ and $A_2 \times Y$ are open and

$$(A_1 \times Y) \cap (A_2 \times Y) = \phi. \tag{9}$$

Hence the set $X \times Y$ is a T_2–space.

● PROBLEM 14–9

Show that the diagonal $D \subset X \times X$ is homeomorphic to X.

SOLUTION:

DEFINITION OF A DIAGONAL

The set

$$D = \{(x, y) : x = y\} \tag{1}$$

is the diagonal of

$$X^2 = X \times X$$

■

Consider the projection

$$\pi_1 : X \times X \to X$$

$$\pi_1 (x, y) = X \tag{2}$$

In Problem 14–4 we proved that the projection is a continuous mapping. Hence, the required homeomorphism is the projection π_1.

$$X \simeq D. \tag{3}$$

Prove the following:

THEOREM

If X is a T_2–space, then the diagonal is closed in $X \times X = X^2$. ■

SOLUTION:

We shall show that $X^2 - D$ is an open set. That is, we have to show that for any point

$$(x, y) \in X^2 - D \tag{1}$$

there are two open sets A and B, such that

$$x \in A, \quad y \in B \tag{2}$$

and

$$A \times B \subset X^2 - D. \tag{3}$$

Since X is a T_2–space and $x \neq y$, there are two open sets A and B, such that

$$x \in A \quad \text{and} \quad y \in B \tag{4}$$

and

$$A \cap B = \phi. \tag{5}$$

Therefore

$$(A \times B) \cap D = \phi \tag{6}$$

that is

$$A \times B \subset X^2 - D. \tag{7}$$

Note that the following is also true: if the diagonal of $X \times X$ is closed, then X is a T_2–space.

● PROBLEM 14–11

1. Show that if

$$f : X \to Y \tag{1}$$

is continuous, then the graph of f

$$G = \{(x, y) : y = f(x)\} \tag{2}$$

is homeomorphic to X.

2. Also show that if Y is a T_2–space, then G is closed in $X \times Y$.

SOLUTION:

1. Consider the mapping

$$h(x) = (x, f(x)). \tag{3}$$

$h(x)$ is a homeomorphism of X onto G. Thus $X \simeq G$.

2. Let us define a mapping

$$p(x, y) = (f(x), y). \tag{4}$$

Then

$$[p(x, y) \in D] \equiv [f(x) = y] \equiv [(x, y) \in G] \tag{5}$$

Thus

$$G = p^{-1}(D). \tag{6}$$

Since p is continuous and D is closed, we conclude that G is closed.

● PROBLEM 14–12

Construct the subbase of the generalized Cartesian product.

SOLUTION:

Consider the family of topological spaces $\{X_\omega : \omega \in \Omega\}$. The generalized Cartesian product is defined by

$$Z = \underset{\omega \in \Omega}{\times} X_\omega \tag{1}$$

The αth coordinate of $f \in Z$, $\pi_\alpha(f)$ is defined by

$$\pi_\alpha(f) = f(\alpha) \tag{2}$$

then

$$\pi_\alpha : Z \to X_\alpha \tag{3}$$

is the projection of Z on X_α.

The topology in Z (called Tychonow topology) is defined as follows:

DEFINITION

A subbase of Z is defined as the family of sets of the form

$$H_{\alpha,G} = \pi_\alpha^{-1}(G) = \{f : f(\alpha) \in G\} \tag{4}$$

where G is open in X_α.

■

Observe that $H_{\alpha, G}$ is the product of G and of all spaces X_β, where $\beta \neq \alpha$.

● PROBLEM 14–13

Show that if $A_\omega \subset X_\omega$ for each $\omega \in \Omega$, then

$$\overline{\underset{\omega}{\times} A_\omega} = \underset{\omega}{\times} \overline{A}_\omega . \tag{1}$$

That is, the Cartesian product of closed sets is always closed.

SOLUTION:

Let $(x_\omega) \in \underset{\omega}{x} X_\omega$ be, such that

$$(x_\omega) \in \overline{\underset{\omega}{\times} A_\omega} . \tag{2}$$

We shall show that for every ω

$$x_\omega \in \overline{A}_\omega \tag{3}$$

and hence

$$(x_\omega) \in \underset{\omega}{\times} \overline{A}_\omega . \tag{4}$$

Let $x_\omega \in U_\omega$, where U_ω is open in X_ω then

$$(x_\omega) \in < U_\omega >. \tag{5}$$

By $< U_\alpha >$ we denote the "slice" of $\underset{\omega}{\times} X_\omega$ where each factor is X_ω except the αth, which is U_α. We have

$$\phi \neq \underset{\omega}{\times} A_\omega \cap < U_\omega > = (U_\omega \cap A_\omega) \times [X\{A_\beta : \beta \neq \omega\}].$$

Hence

$$U_\omega \cap A_\omega \neq \phi \quad \text{and} \quad x_\omega \in \overline{A}_\omega .$$

The converse inclusion is proved in the same manner by reversing the steps.

• PROBLEM 14-14

Let $\{X_\omega : \omega \in \Omega\}$ represent any family of topological spaces. Show that for each fixed $\beta \in \Omega$,

$$\pi_\beta : \underset{\omega}{\times} X_\omega \to X_\beta$$

is continuous and open.

SOLUTION:

Let U denote an open set in X_β. Then

$$\pi_\beta^{-1}(U) = < U >$$

is open in $\underset{\omega}{\times} X_\omega$. Hence π_β is continuous.

Since

$$\pi_\beta < U_{\omega_1}, \ldots, U_{\omega_n} > = \begin{cases} X_\beta & \text{if } \beta \neq \omega_1, \ldots, \omega_n \\ U_\beta & \text{if } \beta = \omega_k \end{cases}$$

the image of any basic open set is open. Hence π_β is open.

• PROBLEM 14-15

Let $\{X_\omega : \omega \in \Omega\}$ denote a family of topological spaces and let

$$f : Y \to \underset{\omega}{\times} X_\omega . \tag{1}$$

Prove that

$$(f \text{ is continuous}) \Leftrightarrow (\pi_\omega \circ f \text{ is continuous for each } \omega \in \Omega).$$

SOLUTION:

$\Rightarrow$ Suppose f is continuous. From 14–14 each π_ω is continuous. Hence

$$\pi_\omega \circ f$$

is continuous for each $\omega \in \Omega$.

$\Leftarrow$ Suppose $\pi_\omega \circ f$ is continuous. Then for each element $<U_\alpha>$ of subbasis in $\underset{\omega}{x} X_\omega$, we have

$$f^{-1} < U_\alpha > = f^{-1}[\pi_\alpha^{-1}(U_\alpha)] = (\pi_\alpha \circ f)^{-1}(U_\alpha).$$

Hence $\times\langle U_\alpha \rangle$ is open and f is continuous.

■

For any

$$f : Y \to \underset{\omega}{\times} X_\omega$$

the map

$$\pi_\alpha \circ f : Y \to X_\alpha$$

is called the αth coordinate function of f.

● PROBLEM 14–16

Let (X_n, d_n) denote a finite or infinite sequence of metric spaces. Define the metric of the product

$$\overset{n}{\underset{k=1}{\times}} X_k \text{ and } \overset{\infty}{\underset{n=1}{\times}} X_n .$$

SOLUTION:

Let $X_1, X_2, \ldots, X_n$ denote a finite sequence of metric spaces. Let x and y denote points in $\overset{n}{\underset{k=1}{\times}} X_k$. Then

$$x = (x_1, x_2, \ldots, x_n)$$

$$y = (y_1, y_2, \ldots, y_n). \tag{1}$$

The distance between x and y can be defined as follows:

$$d(x,y) = \sqrt{\sum_{k=1}^{n} d_k(x_k, y_k)} . \tag{2}$$

It is easy to show that (2) defines a metric in the Cartesian product $X_1 \times X_2 \times \ldots \times X_n$.

Consider the infinite Cartesian product $X_1 \times X_2 \times \ldots \times X_n \times \ldots$. The distance between points

$$x = (x_1, x_2, \ldots) \quad \text{and} \quad y = (y_1, y_2, \ldots)$$

can be defined by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n). \tag{3}$$

Note that d_n is the metric in X_n.

• PROBLEM 14-17

In Problem 14–16, we defined the product $\times X_n$ of metric spaces and the distance. Hence $\left(\overset{\infty}{\underset{n=1}{\times}} X_n, d\right)$ is a metric space.

Metric d induces topology T_d on $\overset{\infty}{\underset{n=1}{\times}} X_n$. In Problem 14–12 we defined the Tychonow topology, T, as $\overset{\infty}{\underset{n=1}{\times}} X_n$. What is the relation between these topologies? (i.e., which one is weaker, or are they the same).

SOLUTION:

The following theorem answers this question.

THEOREM

A set $A \subset X_1 \times X_2 \times \ldots$ is open in the metric sense if, and only if, A is open in the Tychonow topology.

■

Hence the topology induced by the metric agrees with the Tychonow topology.

CHAPTER 15

COUNTABILITY PROPERTIES

Let (R, T) denote the set of real numbers with topology induced by the absolute value metric. Show that

$$\{B(x, 2) : x \in R\}$$

i.e., the collection of open balls

$$B(x, 2) = \{y : y \in R, |x - y| < 2\}$$

is an open cover of R.

Give an example of an open subcover and an improvement of $\{B(x, 2) : x \in R\}$.

SOLUTION:

DEFINITION OF OPEN COVER

An open cover of a topological space (X, T) is a collection

$$\{A_\alpha : \omega \in \Omega, A_\alpha \in T\}$$

of open subsets of X, such that

$$X = \bigcup_\alpha A_\alpha.$$

■

$\{B_\beta\}$ is an open subcover of $\{A_\alpha\}$, if

$$\{B_\beta\} \subset \{A_\alpha\}.$$

The set $\{B(x, 2) : x \in R\}$ is an open cover of R because

$$R = \bigcup_x B(x, 2).$$

The set

$$\{B(n, 2) : n \text{ is an integer}\}$$

is an open subcover of R.

An open cover $\{D_\gamma\}$ is said to be an improvement of $\{A_\alpha\}$, if for each D_γ, there is A_α such that

$$D_\gamma \subset A_\alpha.$$

Hence, we can choose

$$\{B(x, 1) : x \in R\}$$

as an improvement of $\{B(x, 2) : x \in R\}$.

● PROBLEM 15-2

Show that every open cover of a closed and bounded interval $A = [a, b]$ is reducible to a finite cover.

SOLUTION:

We shall apply Heine-Borel Theorem:

If $A = [a, b]$ is a closed and bounded interval and $\{U_n\}$ is a class of open sets, such that

$$A \subset \bigcup_n U_n,$$

then one can choose a finite number of open sets $U_{n_1}, U_{n_2}, \ldots, U_{n_k}$, such that

$$A \subset U_{n_1} \cup U_{n_2} \cup \ldots \cup U_{n_k}.$$

The conclusion from the above theorem is that every open cover of interval $[a, b]$ is reducible to a finite cover.

● PROBLEM 15-3

Show that every metric space is a first countable space.

SOLUTION:

DEFINITION OF FIRST COUNTABLE SPACE

A topological space (X, T) is called a first countable space if, for every point $a \in X$, a countable class B_a of open sets containing a exists, such that every open set A containing a also contains a member of B_a.

■

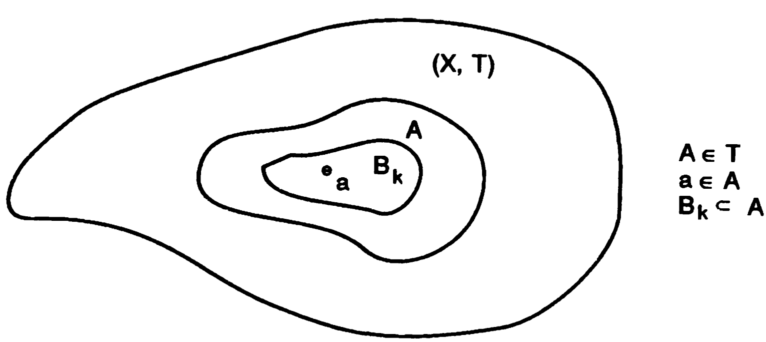

FIGURE 1

That is, a topological space (X, T) is a first countable space iff at every point $a \in X$, a countable local base exists.

Let (X, d) denote a metric space and let $a \in X$ be any point. The set of balls

$$\{B(a, 1), B(a, {}^1/_2), B(a, {}^1/_3), B(a, {}^1/_4), \ldots\}$$

forms a countable local base at $a \in X$.

For every open set A containing a, a ball $B(a, {}^1/_k)$ exists such that

$$a \in B(a, {}^1/_k) \subset A.$$

Hence every metric space is a first countable space.

● PROBLEM 15-4

Show that any subspace (Y, T_Y) of a first countable space (X, T) is also first countable.

SOLUTION:

Let (X, T) be a first countable space and (Y, T_Y) its subspace. Let $a \in Y$.

By hypothesis (X, T) is a first countable space. Hence a countable base B_a in (X, T) exists.

$$B_a = \{B_k : k \in N\}$$

Let us define for each B_k $B'_k = Y \cap B_k$.

Sets B'_k are open in (Y, T_Y) they form a T_Y–local base at $a \in Y$.

The set $\{B'_k : k \in N\}$ is countable. Hence (Y, T_Y) is a first countable space.

● PROBLEM 15-5

Let

$$A_n = \{A_1, A_2, A_3, \ldots\}$$

be a nested local base at $a \in X$. Let $(a_1, a_2, \ldots)$ denote a sequence such that

$$a_1 \in A_1, a_2 \in A_2, \ldots, a_k \in A_k, \ldots.$$

Show that (a_n) converges to a.

SOLUTION:

Let U denote an open set containing a.

Set $\{A_1, A_2, \ldots\}$ is a local base at a. Hence, set A_k exists, such that

$$A_k \subset U.$$

Sets $A_1, A_2, A_3, \ldots$ are nested if

$$A_1 \supset A_2 \supset A_3 \supset A_4 \supset \ldots$$

Hence for any $m > k$ we have

$$A_k \supset A_m$$

and since $a_m \in A_m$ we obtain

$$a_m \in A_k \subset U.$$

Thus

$$a_n \to a.$$

● PROBLEM 15-6

Show that the property of being a first countable space is a topological property.

SOLUTION:

Suppose topological spaces (X, T) and (Y, T_1) are homeomorphic

$$X \simeq Y.$$

We shall show that if (X, T) is a first countable space, then (Y, T_1) is also a first countable space.

Let

$$f : X \to Y$$

be a homeomorphism and let (X, T) be a first countable space. Let $y \in Y$ and V be an open subset of Y, such that $y \in V$. (See Figure 1).

Since f is a homeomorphism, $f^{-1}(y)$ is a point in X and $f^{-1}(V)$ is an open set in X such that

$$f^{-1}(y) \in f^{-1}(V).$$

(X, T) is a first countable space, hence $x = f^{-1}(y)$ has a countable local base B_x. An open set $B \in B_x$ exists, such that

$$x = f^{-1}(y) \in B \subset f^{-1}(V).$$

Thus

$$y \in f(B) \subset V$$

where $f(B)$ is open in (Y, T').

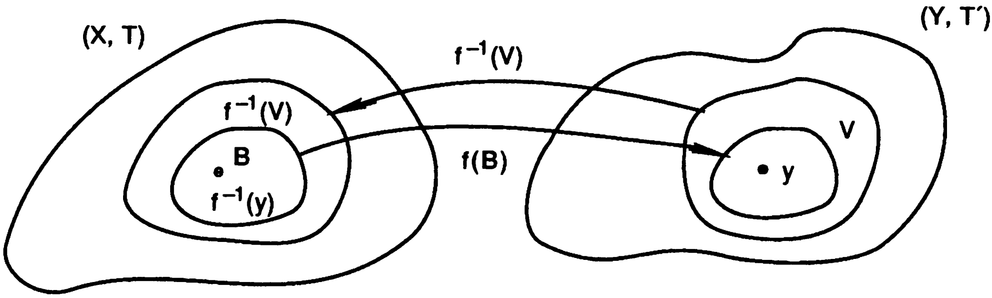

FIGURE 1

● PROBLEM 15-7

Show that if

$$A_x = \{A_1, A_2, A_3, \ldots\}$$

is a countable local base at $x \in X$, then a nested local base exists at x.

SOLUTION:

Let

$$B_1 = A_1, B_2 = A_1 \cap A_2, \ldots, B_n = A_1 \cap \ldots \cap A_n.$$

Each of the sets $B_1, B_2, \ldots$ is open and contains x. Also

$$B_1 \supset B_2 \supset B_3 \supset \ldots$$

Now, let U be an open set containing x, then $k \in N$ exists, such that

$$B_k \subset A_k \subset U.$$

Hence

$$\{B_1, B_2, B_3, \ldots\}$$

is a nested local base at $x \in X$.

• PROBLEM 15-8

Prove this theorem:

THEOREM

A function defined on a first countable space (X, T) is continuous at $x \in X$ iff it is sequentially continuous at x.

■

SOLUTION:

We shall show that if (X, T) is a first countable space, then

$$f: X \to Y$$

is continuous at $x \in X$, if and only if, for every sequence (a_n) converging to $x \in X$, the sequence $(f(a_n))$ converges to $f(x)$. It has been shown that if f is continuous, then f is sequentially continuous. It remains to be shown that

$$(f \text{ is sequentially continuous}) \Rightarrow (f \text{ is continuous})$$

or equivalent

$$(f \text{ is not continuous}) \Rightarrow (f \text{ is not sequentially continuous}).$$

Let $\{A_1, A_2, \ldots\}$ be a nested local base at $x \in X$.

Suppose f is not continuous. Then, an open set $V \subset Y$ exists, such that

$$f(x) \in V, \quad \text{and for every } n \in N, \ A_n \not\subset f^{-1}(V).$$

Thus, for every $n \in N$, an $a_n \in A_n$ exists, such that $a_n \notin f^{-1}(V)$ and therefore $f(a_n) \notin V$.

According to Problem 15–5, (a_n) converges to x, but

$$f(a_n) \not\to f(x).$$

• PROBLEM 15-9

Show that the set of real numbers R with the cofinite topology is not a first countable space.

SOLUTION:

Cofinite topology T contains ϕ and the complements of finite sets. Suppose (R, T) is a first countable space. Let $A(1) = \{A_1, A_2, A_3, \ldots\}$ be a

countable open local base at $1 \in R$. Each A_n is T–open, hence its complement $R–A_n$ is closed and hence finite.

The set

$$A = \bigcup_n (R - A_n)$$

is the countable union of finite sets. Thus A is countable and R is not countable. A point $a \in R$ exists, such that

$$a \neq 1, \quad a \notin A.$$

We have

$$a \in R - A = R - (\bigcup_n (R - A_n)) = \bigcap_n A_n.$$

Hence $a \in A_n$ for every $n \in N$.

The set $R–\{a\}$ is a T–open as a complement of a finite set.

$$1 \in R - \{a\}$$

because $a \neq 1$.

$A(1)$ is a local base at $1 \in R$. $A_k \in A(1)$ exists, such that

$$A_k \subset R - \{a\}.$$

Hence $a \notin A_k$.

That contradicts the fact that for every $n \in N$, $a \in A_n$.

• PROBLEM 15–10

Show that R^2 with the usual topology is a second countable space.

SOLUTION:

DEFINITION OF A SECOND COUNTABLE SPACE

A topological space (X, T) is called a second countable space, if a countable basis B exists for the topology T.

■

Let us define B as a class of open balls in R^2 with rational radii and centers, whose both coordinates are rational numbers. The Cartesian product of three countable sets is a countable set. Hence, B is a countable set. It is easy to see that B is a basis for the usual topology on R^2.

Hence (R^2, T) is a second countable space.

● PROBLEM 15–11

> 1. Show that (R, T), where R is the set of real numbers and T is the discrete topology, is not the second countable space.
>
> 2. Show that: Any second countable space is first countable.

SOLUTION:

1. It is easy to see that if T is a discrete topology, then its base contains all singleton sets.

Set R, and hence, the class of singleton subsets $\{a\}$ of R, is not countable. R with the discrete topology T is not the second countable space.

2. Suppose (X, T) is the second countable space.

Hence, B is a countable base for (X, T). Let B_x consist of all members of B which contain $x \in X$. Then B_x is a countable local base at $x \in X$. Hence, (X, T) is a first countable space.

● PROBLEM 15–12

> Show that:
>
> 1. Every subspace of a second countable space is second countable.
>
> 2. Any second countable space is separable.

SOLUTION:

1. Let B represent a countable base for the second countable space (X, T).

$$B = \{B_n : n \in N\}$$

Then, for any subspace A

$$B_A = \{A \cap B_n : n \in N\}$$

is a countable base for A. Thus, (A, T_A) is a second countable space.

2. **DEFINITION OF SEPARABLE SPACE**

X is said to be separable if X contains a countable dense subset.

■

Let (X, T) denote a second countable space with a countable base B. For each $B_n : n \in N$ select $x(B_n) \in B_n$.

The set

$$\{x(B_n) : B_n \in B\}$$

is a countable subset of X, which is dense.

Hence, (X, T) is separable.

● PROBLEM 15–13

Prove the following:

THEOREM (Lindelöf)

If A is a subset of a second countable space (X, T), then every open cover of A is reducible to a countable cover.

■

SOLUTION:

Let

$$B = \{B_n : n \in N\}$$

denote a countable base for X and let $H = \{H_\alpha : \alpha \in \Omega\}$ denote an open cover of A.

$$A \subset \cup H_\alpha.$$

Hence, for every $x \in A$, $H_\alpha \in H$ exists, such that $x \in H_\alpha$. Since B is a base for X, for every $x \in A$, $B_x \in B$ exists, such that

$$x \in B_x \subset H_\alpha.$$

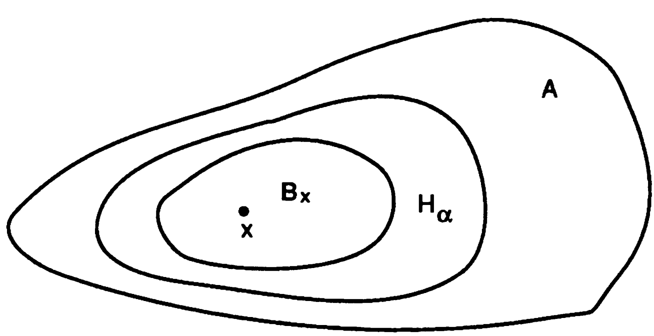

FIGURE 1

Thus

$$A \subset \cup\{B_x : X \in A\}$$

The family of sets $\{B_x : x \in A\}$ is a subset of B, hence it is countable

$$\{B_x : x \in A\} = \{B_k : k \in K\}$$

where K is a countable index set. For each $k \in K$, we can choose one set $H_k \in H$, such that

$$B_k \subset H_k.$$

Thus

$$A \subset \bigcup_k B_k \subset \bigcup_k H_k$$

and $\{H_k : k \in K\}$ is a countable subcover of H.

● PROBLEM 15–14

Show that every second countable space is a Lindelöf space.

SOLUTION:

DEFINITION OF A LINDELÖF SPACE

A topological space (X, T) is called a Lindelöf space, if every open cover of X is reducible to a countable cover.

■

We shall prove the following:

THEOREM

If (X, T) is a second countable space, then every base B for X is reducible to a countable base for X.

■

Let (X, T) represent a second countable space. Then X has a countable base

$$B = \{B_n : n \in N\}.$$

Let $H = \{H_\alpha : \alpha \in \Omega\}$ represent any base for X. Then for each $n \in N$

$$B_n = U\{H_\alpha : H_\alpha \in H_n\}$$

where $H_n \subset H$. Hence H_n is an open cover of B_n.

According to Problem 15–13, H_n is reducible to a countable subcover

H'_n.

$$B_n = U\{H_\alpha : H_\alpha \in H'_n\}, \quad H'_n \subset H_n$$

and H'_n is countable.

Note that

$$H' = \{H_\alpha : H_\alpha \in H'_n, n \in N\}$$

is a base for X, since B is a base. Also

$$H' \subset H$$

and H' are countable.

● PROBLEM 15-15

> Show that every closed subspace of a Lindelöf space is Lindelöf.

SOLUTION:

Suppose A is a closed subspace of a Lindelöf space (X, T) and $\{H_k\}$ is an open cover of A, $k \in K$. Each H_k is open in A,

$$H_k = A \cap V_k$$

where V_k is open in X.

Since A is closed, $X - A$ is open in X. Hence,

$$\{X - A\} \cup \{V_k : k \in K\}$$

is an open cover of X.

There is a countable subcover

$$\{X - A\} \cup \{V_{k_n} : n = 1, 2, 3, \ldots\}$$

of

$$\{X - A\} \cup \{V_k : k \in K\}.$$

Then

$$\{V_{k_n} : n = 1, 2, 3, \ldots\}$$

is a countable open cover of A. Therefore

$$\{A \cap V_{k_n} : n = 1, 2, 3, \ldots\}$$

is a countable open subcover of $\{H_k : k \in \mathrm{K}\}$. Hence, A is a Lindelöf space.

Show that a discrete space X is Lindelöf, if and only if, X is a countable set.

SOLUTION:

Let (X, T) denote a discrete topological space, i.e., T consists of all the subsets of X.

Suppose (X, T) is Lindelöf. Then every open cover of X has a countable subcover. Each singleton set $\{x\}$, $x \in X$ is open in discrete topology. The family of sets

$$\{\{x\} : x \in X\}$$

is an open cover of X

$$X = \bigcup_{x \in X} \{x\}.$$

Since X is Lindelöf, a countable subcover exists

$$\{x_1\}, \{x_2\}, \{x_3\}, \ldots.$$

Hence, X is a countable set.

Conversely, if X is a countable set and (X, T) is a discrete space, then X is Lindelöf. Let

$$A = \{A_\alpha : \alpha \in \Omega\}$$

represent an open cover of X.

$$X \subset \bigcup_{\alpha} A_\alpha.$$

For each $x \in X, A_x \in A$ exists, such that

$$x \in A_x$$

since X is countable

$$X = \{x_1, x_2, \ldots\}$$

we can choose

$$x_1 \in A_1, x_2 \in A_2, \ldots$$

Hence, $A_1, A_2, A_3, \ldots$ is an open subcover of A. Thus, (X, T) is Lindelöf.

● PROBLEM 15–17

Show that:

1. The real line R with the usual topology is a separable set.

2. The real line R with the discrete topology is not a separable set.

SOLUTION:

1. Let Q denote the set of rational numbers. We have showed that Q is a countable set. Since Q is dense in R, the set of real numbers R with the usual topology is separable.

2. Let T denote the discrete topology on R. Then every subset of R is both closed and open. Therefore, the only dense set in R is R itself. Set R is not countable. Hence, R with the discrete topology is not a separable space.

● PROBLEM 15–18

Show that if a topological space (X, T) is a second countable space, then (X, T) is separable.

SOLUTION:

Suppose (X, T) is a second countable space. The family of sets

$$B = \{B_n : n \in N\}$$

is a countable base of X. For each $n \in N$, we choose a point a_n, such that

$$a_n \in B_n.$$

The set

$$A = \{a_n : n \in N\}$$

is countable.

We shall show that each point

$$x \in X - A$$

is an accumulation point of A, that is, that $\overline{A} = X$.

Let $x \in X - A$ and let U represent an open set containing x. At least one set $B_k \in B$ exists, such that

$$a_k \in B_k \subset U.$$

Since $a_k \in A$ and $x \in X - A$,

$$a_k \neq x.$$

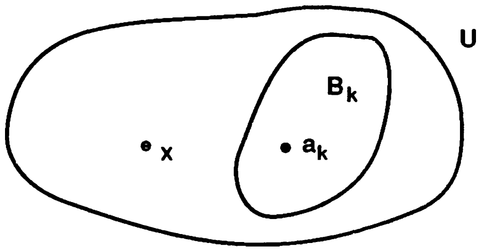

FIGURE 1

We conclude that x is an accumulation point of A, since every open set U containing x also contains a point of A different from x.

● PROBLEM 15–19

1. Show that a discrete space X is separable, if and only if, X is countable.

2. Show that (X, T) with the cofinite topology T is separable.

SOLUTION:

1. Let (X, T) denote a discrete space. Then every subset of X is both open and closed. Therefore, the only dense subset of X is X itself. Hence, X contains a countable dense subset, if and only if, X is countable. Thus, a discrete space X is separable iff X is countable.

2. Suppose X is countable. Then, X is a countable dense subset of (X, T).

Now, suppose, X is not countable. Then X contains non-finite, countable, subset A.

In the cofinite topology, the only closed sets are the finite sets and X. Hence, the closure of A is the entire space X, that is

$$\overline{A} = X.$$

Since A is countable, (X, T) is separable.

● PROBLEM 15–20

Show that (R^2, T) is separable, where R^2 is the plane and T is the topology generated by the half-open rectangles.

$$]a, b] \times]c, d] = \{(x, y) : a < x \leq b, c < y \leq d\}$$

SOLUTION:

For any rectangle $]a, b] \times]c, d]$ a point (x_0, y_0) exists with rational coordinates, such that

$$(x_0, y_0) \in]a, b] \times]c, d] .$$

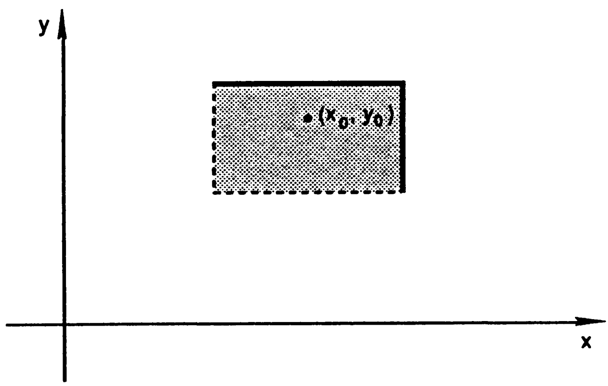

FIGURE 1

Consider the set $A = Q \times Q$ consisting of all points in R^2 with rational coordinates.

Since Q is countable, so is A. The set A is dense in R^2. Hence, (R^2, T) is separable.

● PROBLEM 15–21

Show that any open subspace of a separable space is separable.

SOLUTION:

Let (X, T) represent a separable space and let

$$\{x_n : n \in N\}$$

represent a countable dense subset of (X, T).

We shall show that any open subspace U of X is separable. We define

$$B = U \cap \{x_n : n \in N\}.$$

Let V represent any subset of U which is open in U. Hence V is open in X, since U is open in X. Therefore, V contains some x_n, and hence, a point of B.

We shall apply the following theorem:

THEOREM

A subset A of a topological space (X, T) is dense, if and only if, every open subset of X contains some point of A.

■

Therefore, B is dense in U. Set U contains a countable dense subset B, and hence, U is separable.

● PROBLEM 15–22

Let $B(x, r)$ represent an open sphere in a metric space (X, d), and let

$$d(x, y) < {}^1\!/_3\, r. \tag{1}$$

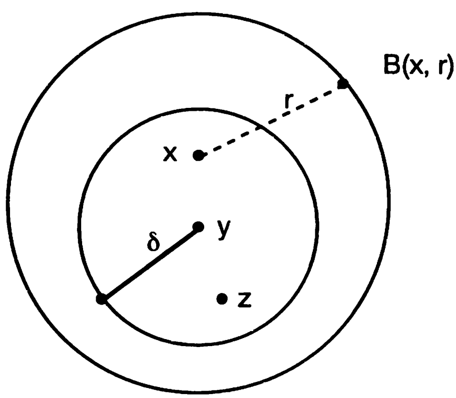

FIGURE 1

Show that if ${}^1\!/_3\, r < \delta < {}^2\!/_3\, r$, then

$$x \in B(y, \delta) \subset B(x, r). \tag{2}$$

SOLUTION:

From (1) and condition ${}^1\!/_3\, r < \delta$ we obtain

$$d(x, y) < \tfrac{1}{3} r < \delta. \tag{3}$$

Hence

$$x \in B(y, \delta). \tag{4}$$

Suppose

$$z \in B(y, \delta) \tag{5}$$

then

$$d(y, z) < \delta. \tag{6}$$

According to triangle inequality

$$d(z, x) < d(y, z) + d(y, x). \tag{7}$$

From (7), (6), (1) we obtain

$$d(z, x) < d(y, z) + d(y, x) < \delta + \tfrac{1}{3} r < \tfrac{2}{3} r + \tfrac{1}{3} r = r. \tag{8}$$

Hence

$$z \in B(x, r)$$

and

$$B(y, \delta) \subset B(x, r). \tag{9}$$

● PROBLEM 15–23

THEOREM

Every separable metric space is second countable. ■

Prove it.

SOLUTION:

Let (X, d) denote a separable metric space. Then A is a countable dense subset of X

$$\overline{A} = \mathrm{X}.$$

According to S we denote the class of all open spheres with centers in A and with rational radii, that is,

$$S = \{B(a, r) : a \in A, r \in Q\}. \tag{1}$$

Since A and Q are countable sets, S is countable.

We will show that S is a base for the topology on X. That is, that for every open set $U \subset X$ and for every $x \in U$

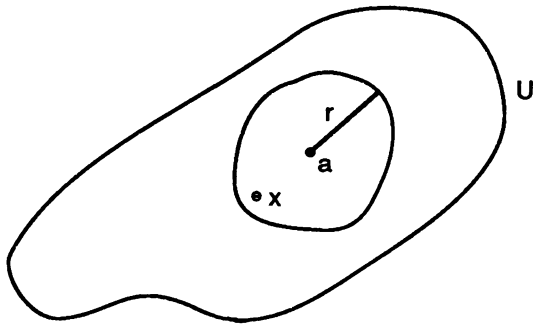

FIGURE 1

there is a sphere $B(a, r) \in S$, such that

$$x \in B(a, r) \subset U. \tag{2}$$

Since $x \in U$, there is an open sphere $B(x, r_0)$, such that

$$x \in B(x, r_0) \subset U. \tag{3}$$

Since A is dense in X, we can always find $a_0 \in A$, such that

$$d(x, a_0) \leq {}^1\!/_3\, r_0. \tag{4}$$

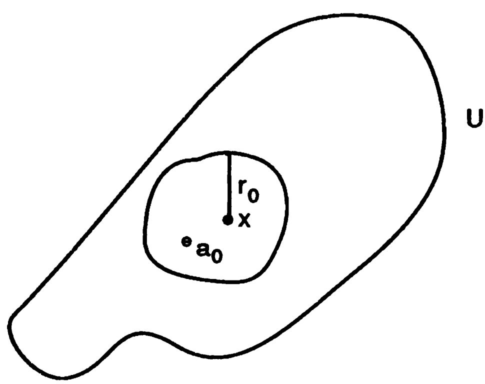

FIGURE 2

Let r_1 denote a rational number, such that

$${}^1\!/_3\, r_0 < r_1 < {}^2\!/_3\, r_0. \tag{5}$$

Then, according to Problem 15–22, we have

$$x \in B(a_0, r_1) \subset B(x, r_0) \subset U. \tag{6}$$

Since $B(a_0, r_1) \in S$ and S are countable, S is a countable base for the topology on X.

• PROBLEM 15–24

Show that the linear space $C[0, 1]$ of all continuous functions on the closed interval $[0, 1]$ with the norm defined by

$$\|f\| = \sup\{|f(x)| : x \in [0, 1]\}$$

is second countable.

SOLUTION:

Let $f \in C[0, 1]$. Then, according to the Weierstrass approximation theorem for any $\varepsilon > 0$, a polynomial $P(x)$ exists with rational coefficients, such that

$$\|f(x) - P(x)\| < \varepsilon$$

that is, for all $x \in [0, 1]$

$$|f(x) - P(x)| < \varepsilon.$$

Hence, the collection of polynomials with rational coefficients is dense in $C[0, 1]$.

The set P of all polynomials with rational coefficients is countable. Hence, $C[0, 1]$ contains a countable dense set, i.e., $C[0, 1]$ which is separable. Since each separable metric space (see Problem 15–23) is second countable, $C[0, 1]$ is second countable.

• PROBLEM 15–25

Show that every Lindelöf metric space is separable.

SOLUTION:

Suppose (X, d) is a Lindelöf space. Let $\alpha > 0$ denote any real positive number, and let A denote a maximal subset of X such that

$$d(a, b) \geq \alpha \quad \text{for every } a, b \in A. \tag{1}$$

The existence of such maximal subset is guaranteed according to Zorn's lemma.

For each $a \in A$ consider $B(a, {}^{\alpha}/_{2})$ and

$$D = X - U\{\overline{B(a, {}^{\alpha}/_{4})} : a \in A\} \tag{2}$$

D is an open set.

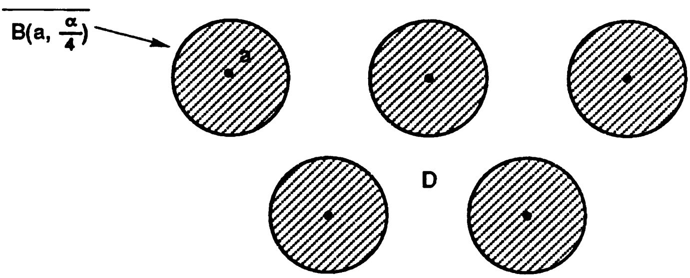

FIGURE 1

Note that

$$\{D\} \cup \{B(a, {}^{\alpha}/_{2}) : a \in A\} \tag{3}$$

is a covering of X by open sets. Since X is Lindelöf, a countable subcovering exists.

Suppose we remove $B(a, {}^{\alpha}/_{2})$ from the original covering (3) for any $a \in A$, then the remaining sets would not cover X (none of them would contain a). Thus, $\{B(a, {}^{\alpha}/_{2}) : a \in A\}$ is countable, hence, A is countable.

We can repeat the above construction for each $\alpha = {}^{1}/_{n}$; $n = 1, 2, 3, \ldots$ and obtain corresponding maximal subsets $\{A_n\}$.

Let

$$P = \bigcup_{n \in N} \{A_n\} \tag{4}$$

P is countable as the union of countably many countable sets. We shall show that P is dense in X.

Suppose $x \in X$ and $\beta > 0$. We have to show that there is $y \in P$, such that $d(y, x) < \beta$.

Set

$$n > {}^{1}/_{\beta}. \tag{5}$$

Then there is $y \in A_n$, such that

$$d(y, x) < \beta. \tag{6}$$

Let U represent any nonempty open subset of X. We choose $x \in U$ and $\beta > 0$, such that

$$B(x, \beta) \subset U.$$

Then $B(x, \beta)$, and hence U, contains an element of P. Therefore P is dense in X. Space (X, d) is separable.

● PROBLEM 15-26

Prove the following important theorem:

THEOREM

If (X, d) is a metric space, then the following statements are equivalent:

1. X is separable
2. X is second countable
3. X is Lindelöf.

■

SOLUTION:

In Problem 15–12, we proved that any second countable space is separable, i.e., we proved

$$(2) \Rightarrow (1) \tag{1}$$

In Problem 15–23, we proved that every separable metric space is second countable, i.e.,

$$(1) \Rightarrow (2) \tag{2}$$

Hence

$$(1) \equiv (2).$$

In Problem 15–14, we proved that every second countable space is Lindelöf, i.e.,

$$(2) \Rightarrow (3). \tag{3}$$

In Problem 15–25, we proved that every Lindelöf metric space is separable, i.e.,

$$(3) \Rightarrow (1) \tag{4}$$

From Equations (4) and (2) we obtain

$$(3) \Rightarrow (2). \tag{5}$$

Therefore

$$(1) \equiv (2) \equiv (3). \tag{6}$$